Particle Propagator of Spin Calogero-Sutherland Model

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Abstract. We obtained the particle propagator for spin 1/2 Calogero-Sutherland model for the finite system and in thermodynamic limit by Uglov's method, the details of which are explained in the previous paper about the hole propagator of this model. Corresponding spectral functions in a full energy-momentum space are calculated numerically and the positions of the divergences of intensity are determined.

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1. Introduction

The exact correlation functions of Calogero-Sutherland model has been studied since its original discovery of the integrability by Sutherland[1, 2, 3, 4, 5, 6, 7, 8]. However, the extension of the calculation of exact correlation functions to arbitrary integer or rational interaction parameters has been realized with use of the relations of Jack polynomial [9, 10, 11, 12, 13, 14, 15]. The Calogero-Sutherland model with the spin degrees of freedom is also known to be integrable, and its eigenfunctions are written in two ways, the Jack polynomials with prescribed symmetry[16, 17] and Yangian Gelfand-Zetlin basis[18, 19]. With use of the former polynomials and the relations of non-symmetric Jack polynomials, the hole propagator[20, 21] and the density correlation function[22] has been obtained. On the other hand, the density correlation function and the spin correlation function has been obtained with use of the Yangian Gelfand-Zetlin basis and the isomorphism to the gl_2-Jack polynomial [23, 24].

In the previous paper[25, 26], we have demonstrated the way to calculate dynamical single-particle Green's functions of the spin Calogero-Sutherland model with the method using the Yangian Gelfand-Zetlin basis and isomorphism, the Uglov's method. Using this scheme, we have obtained the hole propagator. The entire set of single-particle Green's functions is completed by calculating the particle propagator,

$$G^+(x,t) = \frac{\langle \mathrm{g}, N|\psi_\sigma(x,t)\psi_\sigma^\dagger(0,0)|\mathrm{g}, N\rangle}{\langle \mathrm{g}, N|\mathrm{g}, N\rangle} \quad \sigma = \uparrow, \downarrow . \tag{1}$$

with respect to the ground state $|\mathrm{g}, N\rangle$ of N particle systems.

The exact calculation of (1) requires the expansion relations of the field annihilation operator action on eigenfunctions, that can be derived from those on the Macdonald symmetric polynomials. This is why we choose the Uglov's method even though the hole propagator of spin Calogero-Sutherland model has been already obtained by another method[20, 21].

In the present paper, we derive the exact expression of the particle propagator of the spin 1/2 Calogero-Sutherland model for the finite-size system and thermodynamic limit, and we also examine the characteristic features of the spectral function. In section 2, we briefly review the fundamental properties of the spin Calogero-Sutherland model and the method used in this paper. The details of this section is covered in our previous paper[25]. The particle propagator of finite-size system is obtained in section 3, and that of the thermodynamic limit in section 4. The detailed calculations of these two sections are included in two Appendices. In section 5, the spectral function of the particle propagator is drawn in an energy-momentum space.

2. Preliminaries

2.1. Model and Yangian Gelfand-Zetlin basis

Spin 1/2 Calogero-Sutherland model is defined by the Hamiltonian

$$\mathcal{H} = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \frac{2\pi^2}{L^2} \sum_{i<j} \frac{\lambda(\lambda + P_{ij})}{\sin^2[(x_i - x_j)\pi/L]} \tag{2}$$

with the periodic boundary condition, where $0 \leq x_i \leq L$, $\lambda \in \mathbf{Z}_{\geq 0}$, and P_{ij} is spin exchange operator between two particles. We set particles to be bosons (fermions) for odd (even) λ. Introducing the variables $z_i = \exp[2\pi \mathrm{i} x_i/L]$, the exact wave function can be written by a product of Jastrow-type wave function

$$\Psi_{0,N}(z) = \prod_{i=1}^{N} z_i^{-\lambda(N-1)/2} \prod_{i<j} (z_i - z_j)^{\lambda} \tag{3}$$

with $z = (z_1, \cdots, z_N)$ and the Yangian Gelfand-Zetlin basis $\Phi_{\kappa,\alpha}(z, \sigma)$[19, 23, 25]. Here κ belongs to

$$\begin{cases} \mathcal{L}_{N,2} & \lambda \text{ even or } N \text{ odd} \\ \mathcal{L}'_{N,2} & \lambda \text{ odd} \end{cases}$$

with

$$\mathcal{L}_N = \left\{ \kappa \in \mathbf{Z}^N \middle| \, \kappa_i \geq \kappa_{i+1} \text{ for } i \in [1, N-1] \right\} \tag{4}$$

$$\mathcal{L}_{N,2} = \{ \kappa \in \mathcal{L}_N | \, \forall s \in \mathbf{Z}, \sharp\{\kappa_i \,|\, \kappa_i = s\} \leq 2 \} \tag{5}$$

$$\mathcal{L}'_{N,2} = \{ \kappa \,|\, \kappa + 1/2 \in \mathcal{L}_{N,2} \} \,. \tag{6}$$

The subscript $\alpha = (\alpha_1, \cdots, \alpha_N) \in W_\kappa$ with

$$W_\kappa = \left\{ \alpha = (\alpha_1, \cdots, \alpha_N) \in [1, 2]^N \middle| \, \alpha_i < \alpha_{i+1} \text{ if } \kappa_i = \kappa_{i+1} \right\}$$

denotes the spin state of N particles. When $\alpha_i = 1(2)$, we regard the spin of the ith particle as $1/2(-1/2)$. The eigenvalue of S^z_{tot} of the state $\Phi_{\kappa,\alpha}\Psi_{0,N}$ is given by

$$S^z_{\mathrm{tot}} = \sum_{i=1}^{N} (3/2 - \alpha_i).$$

In the following, we set the number of particles N of the ground state to be twice the odd (even) number for even (odd) λ to avoid the degeneracy of the ground state.

The particle propagator (1) is rewritten by Yangian Gelfand-Zetlin basis and the norm defined for them.

The complete set of the state vectors with $N+1$ particles is inserted between the two operators in the numerator of the right-hand side of (1), and the numerator is rewritten in terms of the Yangian Gelfand-Zetlin basis with use of (15). The particle propagator (1) is written as

$$G^+(x, t) = \sum_{\kappa \in \mathcal{L}_{N,2}} \sum_{\alpha \in W_\kappa} \frac{|\langle \mathrm{g}, N | \psi_\downarrow(0,0) | (\kappa, \alpha), N+1 \rangle|^2 e^{-\mathrm{i}\omega_{\kappa,\alpha} t + \mathrm{i} P_{\kappa,\alpha} x}}{\langle (\kappa, \alpha), N+1 | (\kappa, \alpha), N+1 \rangle \cdot \langle \mathrm{g}, N | \mathrm{g}, N \rangle} \tag{7}$$

Here $|(\kappa,\alpha),N+1\rangle$ is the state vector whose wavefunction is given by $\Phi_{\kappa,\alpha}(z,\sigma)\Psi_{0,N+1}$. $\omega_{\kappa,\alpha}$ and $P_{\kappa,\alpha}$ are given, respectively, by

$$\left(\begin{array}{l} \omega_{\kappa,\alpha} = E_{N+1}(\kappa) - E_N(\mathrm{g}) \\ P_{\kappa,\alpha} = (2\pi/L)\sum_{i=1}^{N+1}\kappa_i \end{array}\right. , \tag{8}$$

$$E_{N+1}(\kappa) = (2\pi/L)^2\sum_{i=1}^{N+1}(\kappa_i + \lambda(N+2-2i)/2)^2, \tag{9}$$

$$E_N(\mathrm{g}) = (2\pi/L)^2\sum_{i=1}^{N}(\kappa_i^0 + \lambda(N+1-2i)/2)^2, \tag{10}$$

with $\kappa^0 = (N/4-1/2,\cdots,-N/4+1/2)$.

2.2. Uglov mapping

In Uglov's method[25, 23], the mapping of Yangian Gelfand-Zetlin basis is introduced to calculate the exact dynamical correlation function of spin 1/2 Calogero-Sutherland model, by which the eigenfunctions are transformed to polynomials called gl_2-Jack polynomials $P_\nu^{(\lambda')}(z)$. The mapping $\Omega : \Phi_{\kappa,\alpha}(z,\sigma) \to P_\nu^{(\lambda')}(z)$ with

$$\nu_i = \alpha_{N+1-i} - 2\kappa_{N+1-i} - N + i + K \tag{11}$$

with an even integer K has the following properties[25, 23, 24]:

(i) The mapping conserves the norm, that is,

$$\langle\Phi',\Phi\rangle_{N,\lambda} = \{\Omega(\Phi'),\Omega(\Phi)\}_{N,\lambda}, \tag{12}$$

where on both sides of (12) the norms are defined respectively as

$$\langle f,g\rangle_{N,\lambda} = \frac{1}{N!}\left[\prod_{i=1}^{N}\oint\frac{\mathrm{d}z_i}{2\pi\mathrm{i}z_i}\sum_{\sigma_i}\right]\prod_{i\neq j}\left(1-\frac{z_i}{z_j}\right)^{\lambda} f(z,\sigma)g(z,\sigma), \tag{13}$$

$$\{f,g\}_{N,\lambda} = \frac{1}{N!}\left[\prod_{i=1}^{N}\oint\frac{\mathrm{d}z_i}{2\pi\mathrm{i}z_i}\right]\prod_{i\neq j}\left(1-\frac{z_i}{z_j}\right)^{\lambda+1}\left(1+\frac{z_i}{z_j}\right)^{\lambda} f(z)g(z). \tag{14}$$

Here we note that the inner product between the state vectors of spin 1/2 Calogero-Sutherland model $\langle\Psi'|\Psi\rangle$ is related to $\langle\Phi',\Phi\rangle_{N,\lambda}$ by

$$\langle\Psi'|\Psi\rangle = N!L^N\langle\Phi',\Phi\rangle_{N,\lambda} \tag{15}$$

via $\Psi = \Psi_{0,N}\Phi$ and $\Psi' = \Psi_{0,N}\Phi'$.

(ii) For any symmetric function $f(z)$,

$$\Omega(f(z_1,\cdots,z_N)\Phi_{\kappa,\alpha}) = f(z_1^{-2},\cdots,z_N^{-2})\Omega(\Phi_{\kappa,\alpha}). \tag{16}$$

(iii) gl_2-Jack polynomials are the limit of Macdonald polynomials with $P(z;q,t)$ by $q=-p, t=-p^{\lambda'}, p\to 1$.

2.3. Mapping of particle propagator

In replacing the inner product of the state vectors by the norms of the Yangian Gelfand-Zetlin basis, the field operators are made to act on the eigenfunctions of excited states, in order to deal with the field annihilation operators instead of the field creation operators. The matrix element is then transformed to the norms of gl_2-Jack polynomials in the same manner as the preceding paper. In general, the field annihilation operator of spinless particle and that of spin 1/2 particle act on a wave function $f(\{x\})$ and $f(\{x\},\{\sigma\})$, respectively, as

$$\psi(x)f(x_1,\cdots,x_N) = \sqrt{N}\xi^{N-1}f(x_1,\cdots,x_{N-1},x), \tag{17}$$

$$\psi_\sigma(x)f(x_1,\sigma_1,\cdots,x_N,\sigma_N) = \sqrt{N}\xi^{N-1}f(x_1,\sigma_1,\cdots,x_{N-1},\sigma_{N-1},x,\sigma), \tag{18}$$

where $\xi = 1(-1)$ for bosons (fermions). In term of $\tilde{\psi} = \tilde{\Psi}_{0,N}^{-1}\psi\tilde{\Psi}_{0,N+1}$ with

$$\tilde{\Psi}_{0,N}(z) = \prod_{i=1}^{N} z_i^{-\lambda'(N-1)/2}\prod_{i<j}(z_i - z_j)^{\lambda+1}(z_i + z_j)^{\lambda}, \tag{19}$$

the particle propagator is written as

$$G^+(x,t) = \frac{1}{L(N+1)}\sum_{\nu\in\mathcal{L}_N,S^z_{\mathrm{tot}}=-1/2} e^{-\mathrm{i}\omega_\nu t+\mathrm{i}P_\nu x}\frac{|\{P^{(\lambda')}_{\mathrm{g}-1/2},\tilde{\psi}(0,0)P^{(\lambda')}_\nu\}_{N,\lambda}|^2}{\{P^{(\lambda')}_\nu,P^{(\lambda')}_\nu\}_{N+1,\lambda}\cdot\{1,1\}_{N,\lambda}}, \tag{20}$$

with use of the Uglov mapping with $K = N/2+\lambda-1$ in (11). $P^{(\lambda')}_{\mathrm{g}}(z)$ is the gl_2-Jack polynomial corresponding to the ground state, and given by $\prod_{i=1}^N z_i^{\lambda+1}$ that results from (11) with $K = N/2+\lambda-1$, $\kappa^0 = (N/4-1/2,\cdots,-N/4+1/2)$ and $\alpha^0 = (1,2,\cdots,1,2)$. The expressions for $\tilde{\omega}_\nu = \omega_{\kappa,\alpha}$ and $\tilde{P}_\nu = P_{\kappa,\alpha}$ are obtained as

$$\tilde{\omega}_\nu = (\pi/L)^2\sum_{i=1}^{N+1}(\nu_i + \lambda'(N+1-2i)/2 + \sigma^{\mathrm{p}}_i)^2 - E_N(\mathrm{g}) \tag{21}$$

and

$$\tilde{P}_\nu = -(\pi/L)\sum_{i=1}^{N+1}(\nu_i + \lambda'(N+1-2i)/2 + \sigma^{\mathrm{p}}_i) \tag{22}$$

with $\lambda' = 2\lambda+1$ by substituting

$$\nu_i = \alpha_{N+2-i} - 2\kappa_{N+2-i} - (N+1) + i + (N/2+\lambda-1) \tag{23}$$

$$\sigma^{\mathrm{p}}_i = \frac{3}{2} - \alpha_{N+2-i} \tag{24}$$

into (9) and (8). From (23) and (24), we note that $\sigma^{\mathrm{p}}_i = 1/2(-1/2)$ when $\nu_i - i$ is even (odd). In terms of σ^{p}_i, S^z_{tot} is given by

$$S^z_{\mathrm{tot}} = \sum_{i=1}^{N}\sigma^{\mathrm{p}}_i.$$

2.4. Notations of combinatorial quantities

One of the advantages to use the Uglov mapping lies in the fact that several useful formulae are available in the theory of Macdonald polynomials. Those formulae are expressed in term of the combinatorial quantities related to the Young diagrams. A partition $\nu \in \Lambda_N$ is defined as $\nu \in \mathcal{L}_N$ with $\nu_N \geq 0$ and can be graphically expressed by a Young diagram. and the corresponding Young diagram for a partition ν is denoted by $D(\nu)$, in which the number of squares in i-th row is equal to the i-th element of a partition ν_i. Each square in Young diagram is specified by two-dimensional coordinate with setting upper-left square $s = (1, 1)$. See e.g. Fig.2 of [25]. The first (second) coordinate of a square $s = (i, j)$ represents the vertical (horizontal) axis and increases from top to bottom (from left to right). Let ν'_j be the length of jth columns. Four functions for each square which measure the lengths between the specified square and the edges of the Young diagram are useful to represent formulae specified by partitions, $a(s), a'(s), l(s), l'(s)$ for $s = (i, j)$, as

$$
\begin{aligned}
&a(s) = \nu_i - j, \quad l(s) = \nu'_j - i \\
&a'(s) = j - 1, \quad l'(s) = i - 1.
\end{aligned}
$$

3. Particle propagator in finite size system

3.1. Combinatorial description of particle propagator

In this subsection, we reduce (20) to a combinatorial expression. The part $\tilde{\psi}(0,0)P_\nu^{(\lambda')}$ in the numerator of (20) is decomposed as the product of two factors, one is from gl$_2$-Jack polynomial of $N+1$ particles with one of the variables $P_\nu^{(\lambda')}(z,1)$ fixed, and the other from the ground state wave function of $N+1$ particles and that of N particles transformed by the mapping $\prod_{i=1}^N (z_i - 1)^{\lambda+1}(z_i+1)^\lambda$.

The numerator of the summand in (20) is thus transformed to the norm of two excited states as

$$
\begin{aligned}
&\{P^{(\lambda')}_{\mathrm{g}-1/2}, \tilde{\psi}(0,0)P_\nu^{(\lambda')}\}_{N,\lambda} \\
&= \sqrt{N+1}\xi^N \left\{ \prod_{i=1}^N z_i^{\lambda+1/2}, \prod_{i=1}^N z_i^{-\lambda'/2}(z_i-1)^{\lambda+1}(z_i+1)^\lambda P_\nu^{(\lambda')}(z_1,\cdots,z_N,1) \right\}_{N,\lambda} \\
&= \sqrt{N+1}\xi^N \left\{ \prod_{i=1}^N (1-z_i)^{\lambda+1}(z_i+1)^\lambda, P_\nu^{(\lambda')}(z_1,\cdots,z_N,1) \right\}_{N,\lambda}. \qquad (25)
\end{aligned}
$$

The part $\prod_{i=1}^N (1-z_i)^{\lambda+1}(z_i+1)^\lambda$ is expanded by gl$_2$-Jack polynomials using the formula in [25] by replacing $N-1$ by N as

$$
\prod_{i=1}^N (1-z_i)^{\lambda+1}(1+z_i)^\lambda = \sum_{\substack{\mu \in \Lambda_N \\ \text{s.t. } |C_2(\mu)|+|H_2(\mu)|=|\mu|}} b_\mu P_\mu^{(\lambda')}(z) \qquad (26)
$$

with

$$b_\mu = (-1)^{|\mu|+\sum l'(s)} \frac{\displaystyle\prod_{s\in D(\mu)\setminus C_2(\mu)} (a'(s)-\lambda'(l'(s)+1))}{\displaystyle\prod_{s\in H_2(\mu)} (a(s)+1+\lambda' l(s))}. \tag{27}$$

The set $A \setminus B$ means the complementary set of B in A. Here $C_2(\nu)$ and $H_2(\nu)$ are subsets of $D(\nu)$ defined as

$$C_2(\mu) = \{s \in D(\mu) |\, a'(s) + l'(s) \equiv 0 \mod 2\}, \tag{28}$$

$$H_2(\mu) = \{s \in D(\mu) |\, a(s) + l(s) + 1 \equiv 0 \mod 2\}. \tag{29}$$

An illustration of $C_2(\mu)$ and $H_2(\mu)$ is given in Fig.3 of [25].

On the other hand, $P_\nu^{(\lambda')}(z_1, \cdots, z_N, 1)$ in (25) is expanded by gl$_2$-Jack polynomials $P_\mu^{(\lambda')}(z_1, \cdots, z_N)$ with N variables with use of the corresponding formula that relates the Macdonald polynomial of $N+1$ variables $(z_1, \cdots, z_N, x)$ to that of N variables $(z_1, \cdots, z_N)$[27]

$$P_\nu(z_1, \cdots, z_N, x; q, t) = \sum_{\substack{\mu\in\Lambda_N \\ \text{s.t.}\, \nu/\mu:\text{h.s.}}} \psi_{\nu\mu}(q,t) x^{|\nu|-|\mu|} P_\mu(z_1, \cdots, z_N; q, t), \tag{30}$$

for $\nu \in \Lambda_{N+1}$. Here "h.s." stands for "horizontal strip", and "ν/μ : h.s." means that all the columns of ν and μ satisfy $\nu'_j - \mu'_j = 0$ or 1[27]. The expansion coefficients $\psi_{\nu\mu}(q,t)$ is given by

$$\psi_{\nu\mu}(q,t) = \sum_{i<j} \frac{f(q^{\mu_i-\mu_j} t^{j-i})}{f(q^{\nu_i-\mu_j} t^{j-i})} \frac{f(q^{\nu_i-\mu_{j+1}} t^{j-i})}{f(q^{\mu_i-\nu_{j+1}} t^{j-i})}, \tag{31}$$

$$\left(\begin{array}{l} f(u) = \dfrac{(tu:q)_\infty}{(qu:q)_\infty} \\ (a:q)_\infty = \displaystyle\prod_{r=0}^{\infty} (1-aq^r). \end{array} \right. \tag{32}$$

We express the expansion formula for gl$_2$-Jack polynomials as

$$P_\nu^{(\lambda')}(z_1, \cdots, z_N, 1) = \sum_{\mu\in\mathcal{L}_N} \psi_{\nu\mu}^{(\lambda')} P_\mu^{(\lambda')}(z_1, \cdots, z_N) \tag{33}$$

for $\nu \in \mathcal{L}_{N+1}$. The coefficient $\psi_{\nu\mu}^{(\lambda')}$ for a partition $\nu \in \Lambda_{N+1}$ is obtained by substituting $x = 1$, and taking the limit $q = -p, t = -p^{\lambda'}, p \to 1$ on both sides of (30). For a partition $\nu \in \Lambda_{N+1}$, $\psi_{\nu\mu}^{(\lambda')}$ is given by

$$\psi_{\nu\mu}^{(\lambda')} = \prod_{\substack{s\in H_2(\nu) \\ \text{s.t}\, k\in C_{\nu/\mu}}} \left(\frac{a(s)+1+\lambda' l(s)}{a(s)+\lambda'(l(s)+1)} \right)_\nu \prod_{\substack{s\in H_2(\mu) \\ \text{s.t.}\, k\in C_{\nu/\mu}}} \left(\frac{a(s)+\lambda'(l(s)+1)}{a(s)+1+\lambda' l(s)} \right)_\mu \tag{34}$$

when $\mu \in \Lambda_N$ and ν/μ is a horizontal strip. Otherwise $\psi_{\nu\mu}^{(\lambda')}$ vanishes. Here $C_{\nu/\mu}$ is the set of columns satisfying $\nu'_j - \mu'_j = 0$ in $j \in [\nu_{N+1}, \lambda']$. For $\nu \in \mathcal{L}_{N+1}$ and $\mu \in \mathcal{L}_N$, $\psi_{\nu\mu}^{(\lambda')}$ is given by

$$\psi_{\nu,\mu}^{(\lambda')} = \psi_{\nu-\nu_{N+1},\mu-\nu_{N+1}}^{(\lambda')}. \tag{35}$$

From the above formulae, particle propagator is rewritten as

$$G^{+}(x,t)=\frac{1}{L}\sum_{\substack{\nu\in\mathcal{L}_{N+1}\\ \text{s.t. } S^{z}_{\text{tot}}=-1/2}} e^{-\mathrm{i}\omega_\nu t+\mathrm{i}P_\nu x}\cdot\frac{\left(\sum_{\mu\in\Lambda_N} b_\mu\psi^{(\lambda')}_{\nu\mu}\{P^{(\lambda')}_\mu,P^{(\lambda')}_\mu\}_{N,\lambda}\right)^2}{\{1,1\}_{N,\lambda}\{P^{(\lambda')}_\nu,P^{(\lambda')}_\nu\}_{N+1,\lambda}}. \tag{36}$$

The norms $\{1,1\}_{N,\lambda}$ and $\{P^{(\lambda')}_\mu,P^{(\lambda')}_\mu\}_{N,\lambda}$ are given by

$$\{1,1\}_{N,\lambda}=c_N^{(\lambda',2)},\quad \{P^{(\lambda')}_\mu,P^{(\lambda')}_\mu\}_{N,\lambda}=c_N^{(\lambda',2)}\frac{Y_\mu(1/2)Z(1/(2\lambda'))}{Y_\mu(1/(2\lambda'))Z_\mu(1/2)}. \tag{37}$$

The norm $\{P^{(\lambda')}_\nu,P^{(\lambda')}_\nu\}_{N+1,\lambda}$ for $\nu\in\Lambda_{N+1}$ is given by

$$\{P^{(\lambda')}_\nu,P^{(\lambda')}_\nu\}_{N+1,\lambda}=c_{N+1}^{(\lambda',2)}\frac{Y'_\nu(1)Z_\nu(1/(2\lambda'))}{Y'_\nu((\lambda'+1)/(2\lambda'))Z_\nu(1/2)}. \tag{38}$$

The expressions for $c_N^{(\lambda',2)}$ and $c_{N+1}^{(\lambda',2)}$ are given in Eq.(74) in ref.[25]. $Y_\mu(r)$ and $Y'_\nu(r),Z_\nu(r)$ are defined by

$$Y_\nu(r)\equiv\prod_{s\in C_2(\nu)}\left(\frac{a'(s)}{2\lambda'}+r+\frac{N-1-l'(s)}{2}\right), \tag{39}$$

$$Y'_\nu(r)\equiv\prod_{s\in D(\nu)\backslash C_2(\nu)}\left(\frac{a'(s)}{2\lambda'}+r+\frac{N-1-l'(s)}{2}\right), \tag{40}$$

$$Z_\nu(r)\equiv\prod_{s\in H_2(\nu)}\left(\frac{a(s)}{2\lambda'}+r+\frac{l(s)}{2}\right), \tag{41}$$

respectively. $Y'_\nu(r)$ appears instead of $Y_\nu(r)$ when $2\lambda'\times r$ is even.

3.2. Particle propagator in terms of variables of elementary excitations

In the expression (20), the matrix element is nonzero only for a certain class of the excited states ν. Taking account of the selection rule, the expression of the particle propagator can be reduced so that the elementary excitations become manifest and the thermodynamic limit can be easily taken.

Since the formula (27) contains the expansion over partitions, we only consider $\mu\in\Lambda_N\subset\mathcal{L}_N$. Furthermore in (27) the sum over μ is restricted to the partitions that do not have the square at $s=(1,\lambda'+1)$, that is, μ have at most λ' columns.

Another restriction on μ is from the relation $|C_2(\mu)|+|H_2(\mu)|=|\mu|$, which can be described as the condition that the number of the columns with μ'_j-j : even (we have referred to the set of these columns as Q_μ and the number of them as n_μ in [25]) is equal to λ or $\lambda+1$[25].

The condition $\psi^{(\lambda')}_{\nu\mu}\neq 0$ imposes on $\nu=(\nu_1,\cdots,\nu_N,\nu_{N+1})$ the following restriction:

$$\begin{aligned}&\nu_2\le\nu_1\le\infty,\\ &0\le\nu_{i+1}\le\nu_i\le\nu_{i-1}\le\lambda'\quad\text{for } i\in[2,N],\\ &-\infty\le\nu_{N+1}\le\nu_N.\end{aligned}$$

We classify the excited states ν satisfying (42) into

(i) One left-moving quasi-particle state (0L) specified by ν with

$$\nu_1 > \nu_2 = \cdots = \nu_{N+1} = \lambda'$$

(ii) One right-moving quasi-particle state (0R) specified by ν with

$$\nu_2 = \cdots = \nu_N = 0 > \nu_{N+1}$$

(iii) The states with one-right-moving quasi-particle, one-left-moving quasi-particle and λ' quasi-holes with ν satisfying

$$\nu_1 > \lambda' > \nu_2 \geq \cdots \geq \nu_N \geq 0 > \nu_{N+1}$$

(iv) Other states with ν satisfying

$$\nu_{N+1} \in [0, \lambda'), \quad \text{or } \nu_1 \in (0, \lambda'].$$

The states (iv) does not contribute to the thermodynamic limit and we do not consider them. The contributions from (i)(ii) and (iii) to the particle propagator are denoted, respectively, by $G^{0\mathrm{L}}$, $G^{0\mathrm{R}}$ and G^{+1}, the expressions of which are derived in the following subsections.

3.3. Derivation of $G^{0\mathrm{L}}$ and $G^{0\mathrm{R}}$

First, we derive the expression for $G^{0\mathrm{L}}$ for contribution from one-left-moving quasiparticle state (0L)

$$\nu = (\nu_1, \overbrace{\lambda', \cdots, \lambda'}^{N}). \tag{42}$$

We consider the energy spectrum. From (42) and (9), we obtain

$$\begin{aligned} E_{N+1} = &\left(\frac{\pi}{L}\right)^2 (\nu_1 + \lambda'(N-1)/2 + \sigma_1^{\mathrm{p}})^2 \\ + &\left(\frac{\pi}{L}\right)^2 \sum_{i=2}^{N+1} (\lambda'(N+3-2i)/2 + \sigma_i^{\mathrm{p}})^2 . \end{aligned} \tag{43}$$

By definition of σ_i^{p}, the spin variables in 0L states are given by

$$\sigma_i^{\mathrm{p}} = -1/2, \quad \text{for even } i \in [2, N+1] \tag{44}$$

$$\sigma_i^{\mathrm{p}} = 1/2, \quad \text{for odd } i \in [2, N+1]. \tag{45}$$

The second term of the right-hand side of (43) coincides with the energy of the ground state,

$$E_N(\mathrm{g}) = \left(\frac{\pi}{L}\right)^2 \sum_{i'=1}^{N} (2\kappa_{i',0} + \lambda((N+1) - 2i'))^2 , \tag{46}$$

with

$$\kappa_{2i'-1,0} = \kappa_{2i',0} = (N/2 - i')/2, \quad i' \in [1, N/2], \tag{47}$$

i.e. the summand in (43) for i coincides with the summand in (46) for $i' = N+2-i$. Now we introduce the notations $\tau_{\rm L} = \sigma_1^{\rm p}$ and $\tilde{\rho}_{\rm L} = \rho_{\rm L} + \lambda'(N+1)/2$ with $\rho_{\rm L} = \nu_1 - \lambda'$. We thus obtain

$$E_{N+1} - E_N(\mathrm{g}) = \left(\frac{\pi}{L}\right)^2 (\tilde{\rho}_{\rm L} + \tau_{\rm L})^2 . \tag{48}$$

Similarly, we obtain the expression for momentum for 0L states as

$$P = -\left(\frac{\pi}{L}\right)(\tilde{\rho}_{\rm L} + \tau_{\rm L}) . \tag{49}$$

The 0L states relevant to (36) have the total spin $S_z^{\rm tot} = -1/2$, from which and (44)(45), we set $\tau_{\rm L} = -1/2$.

Next we consider the matrix element in (36). We show that for ν being a (0L) state, the expression (36) reduces to

$$G^+(x,t) \to G^{0\rm L}(x,t) \equiv \frac{1}{L}\frac{c_N^{(\lambda',2)}}{c_{N+1}^{(\lambda',2)}} \sum_{\substack{\nu\in 0\rm L \\ \text{s.t.}\, \tau_{\rm L}=-1/2}} e^{-\mathrm{i}\omega_\nu t + \mathrm{i}P_\nu x} \cdot \frac{Y'_\nu(1/2+\gamma)}{Y'_\nu(1)} \frac{Z_\nu(1/2)}{Z_\nu(\gamma)}, \tag{50}$$

with $\gamma = 1/(2\lambda')$.

For $\nu = (\nu_1, \overbrace{\lambda', \cdots, \lambda'}^{N})$, the only $\mu \in \Lambda_N$ such that $\nu_/\mu$ is horizontal strip is

$$\mu = (\overbrace{\lambda', \cdots, \lambda'}^{N}). \tag{51}$$

For those ν and μ, $\mathrm{C}_{\nu/\mu} = \emptyset$ and hence $\psi_{\nu\mu}^{(\lambda')} = 1$. The expansion coefficient b_μ in (27) for (51) is $(-1)^{N(\lambda+1)} = 1$ because the coefficient of $\prod_{i=1}^N z_i^{\lambda'}$ in the left-hand side of (26) is $(-1)^{N(\lambda+1)}$ and the monomial $\prod_{i=1}^N z_i^{\lambda'}$ in the right-hand side comes only from $P_\mu^{(\lambda')}$ with μ (51). Further, μ (51) is a Galilean shifted partition of $(0,\cdots,0)$ and thus $\left\{P_\mu^{(\lambda')}, P_\mu^{(\lambda')}\right\}_{N,\lambda} = C_N^{(\lambda')}$. From the above consideration, we arrive at (50). The remaining task is the evaluation of the factor $\frac{Y'_\nu(1/2+\gamma)}{Y'_\nu(1)}\frac{Z_\nu(1/2)}{Z_\nu(\gamma)}$, which comes from the norm of (0R) state $\{P_\nu, P_\nu\}_{N+1,\lambda}$ and can be evaluated through $\frac{Y'_{\nu_-}(1/2+\gamma)}{Y'_{\nu_-}(1)}\frac{Z_{\nu_-}(1/2)}{Z_{\nu_-}(\gamma)}$ with $\nu_- = (\nu_1 - \lambda', 0, \cdots, 0) = (\rho_{\rm L}, 0, \cdots, 0)$. In $\nu_- = (\nu_1 - \lambda', 0, \cdots, 0) = (\rho_{\rm L}, 0, \cdots, 0)$, $l'(s) = 0$ and hence $s \in \mathrm{C}_{\nu_-/\mu}(\nu_-)$ is $(1, j)$ for even j. Maximum value of j is $\rho_{\rm L} - 1$. $Y'_{\nu_-}(r)$ is expressed as

$$\begin{aligned} Y'_{\nu_-}(r) &= \prod_{j\in\{2,4,\cdots\}}^{\rho_{\rm L}-1} (\gamma(j-1) + r + (N-1)/2) \\ &= (\lambda')^{-(\rho_{\rm L}-1)} \frac{\Gamma\left(\rho_{\rm L}/2 + \lambda'(r + (N-1)/2)\right)}{\Gamma\left(1/2 + \lambda'(r + (N-1)/2)\right)}. \end{aligned} \tag{52}$$

The cells $s \in H_2(\nu) \cup D(\nu_-)$ are parameterized as

$$s = (1, j), \text{ with } j = 2, 4, \cdots, \rho_{\rm L} - 1$$

With use of this, we obtain

$$Z_{\nu_-}(r) = (\lambda')^{-(\rho_{\rm L}-1)/2}\frac{\Gamma(\rho_{\rm L}/2+\lambda' r)}{\Gamma(1/2+\lambda' r)}. \tag{53}$$

From (52) and (53), it follows that

$$\begin{aligned}\frac{Y'_\nu(1/2+\gamma)}{Y'_\nu(1)}\frac{Z_\nu(1/2)}{Z_\nu(\gamma)}, &= \frac{\Gamma((1+(N+1)\lambda')/2)}{\Gamma(1+N\lambda'/2)\Gamma((1+\lambda')/2)} \\ &\times \frac{\Gamma\left((\rho_{\rm L}+1+\lambda' N)/2)\right)\Gamma\left((\rho_{\rm L}+\lambda')/2)\right)}{\Gamma\left(\rho_{\rm L}+\lambda'(N+1))/2\right)\Gamma\left(\rho_{\rm L}+1)/2\right)}.\end{aligned} \tag{54}$$

From this and (50), we obtain

$$\begin{aligned}&G^{0\rm L}(x,t) \\ &= \frac{K^{(0\rm L)}}{L}\sum_{\rho_{\rm L}=1,3,5,\cdots}\exp\left[-{\rm i}\left(\frac{\pi}{L}\right)^2\left(\tilde\rho_{\rm L}-\frac{1}{2}\right)^2 t - {\rm i}\left(\frac{\pi x}{L}\right)\left(\tilde\rho_{\rm L}-\frac{1}{2}\right)\right]F^{(0\rm L)}\end{aligned} \tag{55}$$

with

$$\begin{aligned}F^{(0\rm L)} &= \frac{\Gamma\left((\rho_{\rm L}+1+\lambda' N)/2)\right)\Gamma\left((\rho_{\rm L}+\lambda')/2)\right)}{\Gamma\left(\rho_{\rm L}+\lambda'(N+1))/2\right)\Gamma\left(\rho_{\rm L}+1)/2\right)} \\ &= \frac{\Gamma\left((\tilde\rho_{\rm L}-\tilde\rho_{\rm R,0}+1)/2)\right)\Gamma\left(\tilde\rho_{\rm L}-\tilde\rho_{\rm L,0})/2\right)}{\Gamma\left((\tilde\rho_{\rm L}-\tilde\rho_{\rm R,0}+\lambda')/2\right)\Gamma\left(\tilde\rho_{\rm L}-\tilde\rho_{\rm L,0}+1-\lambda')/2\right)}\end{aligned} \tag{56}$$

and

$$K^{(0\rm L)} = \frac{c_N^{(\lambda',2)}}{c_{N+1}^{(\lambda',2)}}\frac{\Gamma((1+(N+1)\lambda')/2)}{\Gamma(1+N\lambda'/2)\Gamma((1+\lambda')/2)} = 1. \tag{57}$$

In (56), we introduce the notations

$$\tilde\rho_{\rm L,0} = -\tilde\rho_{\rm R,0} = \lambda'(N-1)/2,$$

for convenience.

The contribution from one-right-moving quasi-particle states (0R) can be derived in a similar way. The (0R) states are specified by

$$\nu = (\overbrace{0,\cdots,0}^{N}, \nu_{N+1}), \tag{58}$$

the spin and the energy spectrum of which are given by

$$\sigma_i^{\rm p} = -1/2, \quad \text{for odd } i\in[1,N-1] \tag{59}$$

$$\sigma_i^{\rm p} = 1/2, \quad \text{for even } i\in[2,N] \tag{60}$$

and

$$\omega_\nu = E_{N+1}-E_N({\rm g}) = \left(\frac{\pi}{L}\right)^2(\tilde\rho_{\rm R}+\tau_{\rm R})^2 \tag{61}$$

$$P = -\frac{\pi}{L}(\tilde\rho_{\rm R}+\tau_{\rm R}) \tag{62}$$

with

$$\tilde{\rho}_{\mathrm{R}} = \nu_{N+1} - \frac{\lambda'(N+1)}{2}, \quad \tau_{\mathrm{R}} = \sigma^{\mathrm{p}}_{N+1}. \tag{63}$$

From the condition $S_z^{\mathrm{tot}} = -1/2$, (59) and (60), the spin of the right-moving quasi-particle is fixed to be $\tau_{\mathrm{R}} = -1/2$ and hence ν_{N+1} is taken to be even. We consider $\psi^{(\lambda')}_{\nu-\nu_{N+1},\mu-\nu_{N+1}}$ instead of $^{(\lambda')}$ because ν is not a partition. For $\nu - \nu_{N+1} = (\overbrace{-\nu_{N+1},\cdots,-\nu_{N+1}}^{N},0)$, the only μ such that $0 \le \mu_i \le \lambda'$ for $i \in [1,N]$ and $(\nu-\nu_{N+1})/(\mu-\nu_{N+1})$ is a horizontal strip is $\mu = 0^N = (\overbrace{0,\cdots,0}^{N})$. $\mathrm{C}_{(\nu-\nu_{N+1})/(\mu-\nu_{N+1})}$ is thus $\emptyset$ and $\psi^{(\lambda')}_{(\nu-\nu_{N+1})/(\mu-\nu_{N+1})} = 1$. Consequently, for ν being a (0R) state, the expression (36) reduces to

$$G^{+}(x,t) \to G^{0\mathrm{R}}(x,t) \equiv \frac{1}{L}\frac{c_N^{(\lambda',2)}}{c_{N+1}^{(\lambda',2)}} \sum_{\substack{\nu\in 0\mathrm{L} \\ \text{s.t. } \sigma^{\mathrm{p}}_{N+1}=-1/2}} e^{-\mathrm{i}\omega_\nu t+\mathrm{i}P_\nu x} \cdot \frac{Y'_\nu(1/2+\gamma)}{Y'_\nu(1)}\frac{Z_\nu(1/2)}{Z_\nu(\gamma)}. \tag{64}$$

For $\nu - \nu_{N+1} = (-\nu_{N+1},\cdots,-\nu_{N+1},0)$, $Y'_\nu(r)$ and $Z'_\nu(r)$ are written, respectively, by

$$\begin{aligned} Y'_\nu(r) = & \prod_{j=1,3,\cdots}^{-\nu_{N+1}-1}\left(\prod_{i=2,4,\cdots}^{N}\left(\gamma(j-1)+r+\frac{N-i}{2}\right)\right) \\ & \times \prod_{j=2,4,\cdots}^{-\nu_{N+1}}\left(\prod_{i=1,3,\cdots}^{N-1}\left(\gamma(j-1)+r+\frac{N-i}{2}\right)\right) \end{aligned} \tag{65}$$

and

$$\begin{aligned} Z_\nu(r) = & \prod_{j=1,3,\cdots}^{-\nu_{N+1}-1}\left(\prod_{i=2,4,\cdots}^{N}\left(\gamma(-\nu_{N+1}-j)+r+\frac{N-i}{2}\right)\right) \\ & \times \prod_{j=2,4,\cdots}^{-\nu_{N+1}}\left(\prod_{i=1,3,\cdots}^{N-1}\left(\gamma(-\nu_{N+1}-j)+r+\frac{N-i}{2}\right)\right) \\ = & \prod_{j'=1,3,\cdots}^{-\nu_{N+1}-1}\left(\prod_{i=1,3,\cdots}^{N-1}\left(\gamma(j'-1)+r+\frac{N-i}{2}\right)\right) \\ & \times \prod_{j'=2,4,\cdots}^{-\nu_{N+1}}\left(\prod_{i=2,4,\cdots}^{N}\left(\gamma(j'-1)+r+\frac{N-i}{2}\right)\right). \end{aligned} \tag{66}$$

In the second equality of (66), we have changed the dummy variables from j to $j' = -\nu_{N+1}+1-j$.

From (65), we obtain

$$\begin{aligned} Y'_\nu(r+1/2) = & \prod_{j'=1,3,\cdots}^{-\nu_{N+1}-1}\left(\prod_{i=1,3,\cdots}^{N-1}\left(\gamma(j'-1)+r+\frac{N-i}{2}\right)\right) \\ & \times \prod_{j'=2,4,\cdots}^{-\nu_{N+1}}\left(\prod_{i=0,2,4,\cdots}^{N-2}\left(\gamma(j'-1)+r+\frac{N-i}{2}\right)\right). \end{aligned} \tag{67}$$

Dividing (67) by (66), we obtain

$$\frac{Y'_\nu(r+1/2)}{Z_\nu(r)} = \prod_{j'=2,4,\cdots}^{-\nu_{N+1}} \frac{\gamma(j-1)+r+N/2}{\gamma(j-1)+r}$$
$$= \frac{\Gamma\left((-\nu_{N+1}+1+\lambda' N)/2+\lambda' r\right)\Gamma\left(1/2+\lambda' r\right)}{\Gamma\left((-\nu_{N+1}+1)/2+\lambda' r\right)\Gamma\left((1+\lambda' N)/2+\lambda' r\right)}, \tag{68}$$

from which

$$\frac{Y'_\nu(1/2+\gamma)}{Y'_\nu(1)}\frac{Z_\nu(1/2)}{Z_\nu(\gamma)}$$
$$= \frac{\Gamma((1+(N+1)\lambda')/2)}{\Gamma(1+N\lambda'/2)\Gamma((1+\lambda')/2)}$$
$$\times \frac{\Gamma\left((-\nu_{N+1}+2+\lambda' N)/2)\right)\Gamma\left((-\nu_{N+1}+1+\lambda')/2)\right)}{\Gamma\left(-\nu_{N+1}+1+\lambda'(N+1))/2\right)\Gamma\left(-\nu_{N+1}+2)/2\right)} \tag{69}$$

follows.@ From this, (61), (62), (64) and the second equality of (57), we arrive at

$$G^{0\mathrm{R}}(x,t)$$
$$= \frac{1}{L}\sum_{\nu_{N+1}=-2,-4,-6,\cdots} \exp\left[-\mathrm{i}\left(\frac{\pi}{L}\right)^2\left(\tilde\rho_\mathrm{R}-\frac{1}{2}\right)^2 t - \mathrm{i}\left(\frac{\pi x}{L}\right)\left(\tilde\rho_\mathrm{R}-\frac{1}{2}\right)\right] F^{(0\mathrm{R})} \tag{70}$$

with

$$F^{(0\mathrm{R})} = \frac{\Gamma\left((-\nu_{N+1}+2+\lambda' N)/2)\right)\Gamma\left((-\nu_{N+1}+1+\lambda')/2)\right)}{\Gamma\left(-\nu_{N+1}+1+\lambda'(N+1))/2\right)\Gamma\left(-\nu_{N+1}+2)/2\right)}.$$
$$= \frac{\Gamma\left((\tilde\rho_{\mathrm{L},0}-\tilde\rho_\mathrm{R}+2)/2)\right)\Gamma\left(\tilde\rho_{\mathrm{R},0}-\tilde\rho_\mathrm{R}+1)/2\right)}{\Gamma\left((\tilde\rho_{\mathrm{L},0}-\tilde\rho_\mathrm{R}+\lambda'+1)/2\right)\Gamma\left(\tilde\rho_{\mathrm{R},0}-\tilde\rho_\mathrm{R}+2-\lambda')/2\right)}.$$

3.4. Derivation of G^{+1}

First we decompose the excitation energy $\tilde\omega_\nu$ (21) into the terms with $i=1,N+1$ and others. The former has been considered in the previous subsection. Accordingly, $\tilde\omega_\nu$ is rewritten as

$$\tilde\omega_\nu = (\pi/L)^2\{(\tilde\rho_\mathrm{L}+\tau_\mathrm{L})^2 + (\tilde\rho_\mathrm{R}+\tau_\mathrm{R})^2\} + \tilde\omega'_\nu \tag{71}$$

with

$$\tilde\omega'_\nu = \left(\frac{\pi}{L}\right)^2 \sum_{i=2}^{N}\left(\nu_i + \frac{\lambda'(N+1-2i)}{2} + \sigma_i^\mathrm{p}\right)^2 - E_N(\mathrm{g}). \tag{72}$$

Introducing a partition $\zeta=(\zeta_1,\cdots,\zeta_N)\in\Lambda_N$ as

$$\zeta_i = \nu_{i+1}, \quad i\in[1,N], \tag{73}$$

the energy (72) becomes

$$\tilde\omega'_\nu = \left(\frac{\pi}{L}\right)^2 \sum_{i=2}^{N}\left(\zeta_i + \frac{\lambda'(N-1-2i)}{2} + \sigma_{i+1}^\mathrm{p}\right)^2 - E_\mathrm{g}. \tag{74}$$

The first term in the right-hand side coincides with (B.2) in [25] when we replace ζ_i by μ_i and $\sigma^{\mathrm{p}}_{i+1}$ by $3/2-\alpha_{N-i}$. Further the relation

$$\sigma^{\mathrm{p}}_{i+1}=\begin{cases}-1/2 & (i,\zeta_i)\in C_2(\zeta)\\ \ \ 1/2 & (i,\zeta_i)\in D(\zeta)\setminus C_2(\zeta)\end{cases}$$

coincides with (B.4) in [25] under the same replacement. Therefore we can rewrite (74) following the argument of Appendix B in [25]. For $j\in[1,\lambda']$, let ζ'_j be the length of jth each column in $D(\zeta)$ and σ_j be "spin variables" defined by

$$\sigma_j=\begin{cases}\ \ 1/2, & \zeta_j-j \text{ is odd}\\ -1/2, & \zeta_j-j \text{ is even.}\end{cases} \tag{75}$$

Furthermore, we introduce the renormalized momentum

$$\check{\zeta}_j=\zeta'_j-\frac{N-1}{2}+\frac{\lambda'+1-j}{2\lambda'}. \tag{76}$$

In terms of (75) and (76), $\tilde{\omega}'_\nu$ is rewritten as

$$\tilde{\omega}'_\nu=\left(\frac{\pi}{L}\right)^2\left[-\lambda'\sum_{j=1}^{\lambda'}\left(\check{\zeta}_j+\sigma_j\right)^2+\frac{(\lambda')^2-2}{3}\right], \tag{77}$$

from which

$$\tilde{\omega}_\nu=\left(\frac{\pi}{L}\right)^2\left[\sum_{i=\mathrm{L,R}}(\tilde{\rho}_i+\tau_i)^2-\lambda'\sum_{j=1}^{\lambda'}\left(\check{\zeta}_j+\sigma_j\right)^2+\frac{(\lambda')^2-2}{3}\right] \tag{78}$$

follows. Similarly, the total momentum is rewritten as

$$\tilde{P}_\nu=-\frac{\pi}{L}\left[\sum_{i=\mathrm{L,R}}(\tilde{\rho}_i+\tau_i)+\sum_{j=1}^{\lambda'}\left(\check{\zeta}_j+\sigma_j\right)\right]. \tag{79}$$

The norm $\{P^{(\lambda')}_\nu,P^{(\lambda')}_\nu\}_{N+1,\lambda}$ can be expressed in terms of rapidities and spins $(\tilde{\rho}_{\mathrm{L}},\tau_{\mathrm{L}},\tilde{\rho}_{\mathrm{R}},\tau_{\mathrm{R}},\check{\zeta}_1,\sigma_1,\cdots,\check{\zeta}_{\lambda'},\sigma_{\lambda'})$ of elementary excitations. The derivation of the norm is given in Appendix A.1.

ν can be decomposed into λ' quasi-holes and two quasi-particles. Definition of spin for quasi-hole is identical to that for μ, substituting ν'_j instead of μ'_j. When $\nu_1\geq\lambda'$ or $\nu_{N+1}\leq 0$, quasi-particles also have momenta and spins by the following rules with variables $\rho_{\mathrm{L}}=\nu_1-\lambda'$ and $\rho_{\mathrm{R}}=-\nu_{N+1}$:

$$\tau_{\mathrm{L}}=\begin{cases}+1/2 & (\rho_{\mathrm{L}}:\text{even})\\ -1/2 & (\rho_{\mathrm{L}}:\text{odd})\end{cases},\quad \tau_{\mathrm{R}}=\begin{cases}+1/2 & (\rho_{\mathrm{R}}:\text{odd})\\ -1/2 & (\rho_{\mathrm{R}}:\text{even})\end{cases}.$$

Since z component of the total spin is described by spin variables of quasi-holes σ_j and quasi-particles $\tau_{\mathrm{L,R}}$, the condition for ν that the total spin of the state ν is $-1/2$ is rewritten as

$$\begin{aligned}S^z_{\mathrm{tot}}&=\sum_{j=1}^{\lambda'}\sigma_j+\tau_{\mathrm{L}}+\tau_{\mathrm{R}}\\ &=n_\nu-\frac{\lambda'}{2}+\tau_{\mathrm{L}}+\tau_{\mathrm{R}}=-\frac{1}{2},\end{aligned} \tag{80}$$

where n_ν is the number of quasi-holes with spin $+1/2$ for the state ν.

We thus obtain particle propagator for finite system in terms of rapidities of quasi-holes and quasi-particles $\tilde{\rho}_{\rm L}, \tilde{\rho}_{\rm R}, \check{\zeta}_j$, and spins of them $\tau_{\rm L}$, $\tau_{\rm R}$ and σ_j. We introduce the auxiliary quantities $\zeta_0 = N-1$, $\zeta_{\lambda'+1} = 0$, $\sigma_0 = 1/2$, $\sigma_{\lambda'+1} = -1/2$. In the exact expression for G^+, matrix element is composed by two parts, L_ν which is the part independent on the sum about μ and M_ν which is the part dependent on it.

$$G^+(x,t) = \sum_{\substack{\nu\in\mathcal{L}_N \\ S^z_{\rm tot}=-1/2}} L_\nu M_\nu^2 e^{-{\rm i}\tilde{\omega}_\nu t + {\rm i}\tilde{P}_\nu x} \tag{81}$$

with

$$L_\nu = \frac{1}{L}\frac{c_N^{(\lambda',2)}}{c_{N+1}^{(\lambda',2)}}\frac{Y'_\nu(1/2+\gamma)}{Y'_\nu(1)}\frac{Z_\nu(1/2)}{Z_\nu(\gamma)}\cdot\left(\frac{X_{\tilde{\nu}}}{Z_{\tilde{\nu}}(1/2)}\frac{Y_{\tilde{\nu}}(1/2)}{Y_{\tilde{\nu}}(\gamma)}\right)^2 \tag{82}$$

$$M_\nu = \sum_{\substack{\mu\in\Lambda_N \\ \text{s.t.}\,\nu/\mu:\text{h.s.} \\ |C_2(\mu)|+|H_2(\mu)|=|\mu|}} (-1)^{|\mu|+\sum_\mu a'(s)}\psi^{(\lambda')}_{\nu\mu}\frac{X_\mu}{Z_\mu(1/2)}\frac{Y_\mu(1/2)}{Y_\mu(\gamma)}\left(\frac{X_{\tilde{\nu}}}{Z_{\tilde{\nu}}(1/2)}\frac{Y_{\tilde{\nu}}(1/2)}{Y_{\tilde{\nu}}(\gamma)}\right)^{-1}. \tag{83}$$

Here $X_\mu = (-1)^{|\mu|-|C_2(\mu)|}Y'_\mu(-N/2)$ and $\tilde{\nu}$ is defined as the partition $\tilde{\nu} = (\nu_1 - \rho_{\rm L}, \nu_2, \cdots, \nu_N, \nu_{N+1}+\rho_{\rm R})$. The details of the calculation of (82) and (83) are written in Appendix A. The result is described by renormalized momenta and spins of quasi-particles and quasi-holes, as

$$\check{\zeta}_j = \zeta_j - \frac{N-1}{2} + \frac{\lambda+1-j}{\lambda'} \tag{84}$$

$$\tilde{\rho}_{\rm L} = \rho_{\rm L} + \frac{\lambda'(N+1)}{2} \tag{85}$$

$$\tilde{\rho}_{\rm R} = -\rho_{\rm R} - \frac{\lambda'(N+1)}{2}. \tag{86}$$

The explicit expression for $G^{+1}(x,t)$ is given by

$$G^{+1}(x,t) = \sum_{\nu\in\mathcal{L}_{N+1}} \delta_{\sum\sigma_j+\tau_{\rm L}+\tau_{\rm R},-1/2} L_\nu M_\nu^2 e^{-{\rm i}\tilde{\omega}_\nu t+{\rm i}\tilde{P}_\nu x} \tag{87}$$

$$\begin{aligned}
L_\nu = &\frac{1}{L}\frac{\Gamma(1)}{\Gamma(\lambda+1)}\frac{X_{\tilde{\nu}}^2}{Z_{\tilde{\nu}}(1/2)Z_{\tilde{\nu}}(\gamma)}\frac{Y_{\tilde{\nu}}(1/2)^2}{Y_{\tilde{\nu}}(\gamma)^2}\frac{Y'_{\tilde{\nu}}(1/2+\gamma)}{Y'_{\tilde{\nu}}(1)} \\
&\times\prod_{j=1}^{\lambda'}\left(\frac{\check{\zeta}_j-\check{\zeta}_{\lambda'+1}}{\check{\zeta}_j-\check{\zeta}_{\lambda'+1}+1-1/\lambda'}\right)^{1-\delta_{\sigma_j\uparrow}}\left(\frac{\tilde{\rho}_{\rm L}+\lambda'\check{\zeta}_j+(\lambda'-1)/2}{\tilde{\rho}_{\rm L}+\lambda'\check{\zeta}_j-(\lambda'-1)/2}\right)^{1-\delta_{\sigma_j\tau_{\rm L}}} \\
&\qquad\times\left(\frac{\check{\zeta}_0-\check{\zeta}_j}{\check{\zeta}_0-\check{\zeta}_j+1-1/\lambda'}\right)^{\delta_{\sigma_j\uparrow}}\left(\frac{\tilde{\rho}_{\rm R}+\lambda'\check{\zeta}_j-(\lambda'-1)/2}{\tilde{\rho}_{\rm R}+\lambda'\check{\zeta}_j+(\lambda'-1)/2}\right)^{1-\delta_{\sigma_j\tau_{\rm R}}} \\
&\times\frac{\Gamma\left(1+\frac{\lambda'}{2}N\right)^2)\Gamma\left(\frac{\lambda'}{2}(N+2)\right)}{\Gamma\left(\frac{1}{2}+\frac{\lambda'}{2}(N+1)\right)^3} \\
&\times\frac{\Gamma\left[\tilde{\rho}_{\rm L}-\tilde{\rho}_{\rm R,0}+\lambda'+\delta_{\tau_{\rm L},\uparrow})/2\right]}{\Gamma\left[\tilde{\rho}_{\rm L}-\tilde{\rho}_{\rm R,0}+1+\delta_{\tau_{\rm L},\uparrow})/2\right]}\frac{\Gamma\left[(\tilde{\rho}_{\rm L}-\tilde{\rho}_{\rm L,0}+\delta_{\tau_{\rm L},\uparrow})/2\right]}{\Gamma\left[(\tilde{\rho}_{\rm L}-\tilde{\rho}_{\rm L,0}+1-\lambda'+\delta_{\tau_{\rm L},\uparrow})/2\right]}
\end{aligned}$$

$$\times \frac{\Gamma\left[(\tilde{\rho}_{\mathrm{L},0} - \tilde{\rho}_{\mathrm{R}} + \lambda' + 1 - \delta_{\tau_{\mathrm{R}},\uparrow})/2\right]}{\Gamma\left[(\tilde{\rho}_{\mathrm{L},0} - \tilde{\rho}_{\mathrm{R}} + 2 - \delta_{\tau_{\mathrm{R}},\uparrow})/2\right]} \frac{\Gamma\left[(\tilde{\rho}_{\mathrm{R},0} - \tilde{\rho}_{\mathrm{R}} + 1 - \delta_{\tau_{\mathrm{R}},\uparrow})/2\right]}{\Gamma\left[(\tilde{\rho}_{\mathrm{R},0} - \tilde{\rho}_{\mathrm{R}} + 2 - \lambda' - \delta_{\tau_{\mathrm{R}},\uparrow})/2\right]}$$
$$\times \frac{\Gamma\left[(\tilde{\rho}_{\mathrm{L}} - \tilde{\rho}_{\mathrm{R}} + 2 - \lambda' - \delta_{\tau_{\mathrm{L}},\tau_{\mathrm{R}}})/2\right]}{\Gamma\left[(\tilde{\rho}_{\mathrm{L}} - \tilde{\rho}_{\mathrm{R}} + 1 - \delta_{\tau_{\mathrm{L}},\tau_{\mathrm{R}}})/2\right]} \frac{\Gamma\left[(\tilde{\rho}_{\mathrm{L}} - \tilde{\rho}_{\mathrm{R}} + 1 + \delta_{\tau_{\mathrm{L}},\tau_{\mathrm{R}}})/2\right]}{\Gamma\left[(\tilde{\rho}_{\mathrm{L}} - \tilde{\rho}_{\mathrm{R}} + \lambda' + \delta_{\tau_{\mathrm{L}},\tau_{\mathrm{R}}})/2\right]}, \tag{88}$$

with

$$X_{\tilde{\nu}} = \prod_{j=1}^{\lambda'} \frac{\Gamma\left[(\check{\zeta}_j - \check{\zeta}_{\lambda'+1} + 2 - \delta_{\sigma_j,\uparrow})/2\right]}{\Gamma\left(j/\lambda'\right)} \tag{89}$$

$$Y_{\tilde{\nu}}(r) = \prod_{j=1}^{\lambda'} \frac{\Gamma\left[\gamma + N/2 + (j-1)/\lambda' + r\right]}{\Gamma\left[(\check{\zeta}_0 - \check{\zeta}_j + 1 - 1/\lambda' - \delta_{\sigma_j\uparrow})/2 + r\right]} \tag{90}$$

$$Y'_{\tilde{\nu}}(r) = \prod_{j=1}^{\lambda'} \frac{\Gamma\left[N/2 + (j-1)/\lambda' + r\right]}{\Gamma\left[(\check{\zeta}_0 - \check{\zeta}_j - 1/\lambda' + \delta_{\sigma_0,\sigma_j})/2 + r\right]} \tag{91}$$

$$Z_{\tilde{\nu}}(r) = \Gamma\left[r + 1/2\right]^{-\lambda'} \prod_{j=1}^{\lambda'} \Gamma[(\check{\zeta}_j - \check{\zeta}_{\lambda'+1} + 1 - 1/\lambda' + \delta_{\sigma_j,\sigma_{\lambda'+1}})/2 + r]$$
$$\times \prod_{1 \le j < k \le \lambda'} \frac{\Gamma\left[(\check{\zeta}_j - \check{\zeta}_k + 1 - 1/\lambda' - \delta_{\sigma_j,\sigma_k})/2 + r\right]}{\Gamma\left[(\check{\zeta}_j - \check{\zeta}_k + \delta_{\sigma_j,\sigma_k})/2 + r\right]}, \tag{92}$$

and

$$M_\nu = \prod_{j=1}^{\lambda'} \left(\frac{\check{\zeta}_j - \check{\zeta}_{\lambda'+1} + 1 - 1/\lambda'}{\check{\zeta}_j - \check{\zeta}_{\lambda'+1}} \right)^{1-\delta_{\sigma_j\uparrow}}$$
$$\times \sum_{\substack{\mu \in \Lambda_N \\ \text{s.t.}\nu/\mu:\text{h.s.} \\ n_\mu = \lambda, \lambda+1,\ \mu_1 \le \lambda'}} (-1)^{|\mathrm{odd} \cap I|} \prod_{j \in J} \left(\frac{\tilde{\rho}_{\mathrm{L}} + \lambda' \check{\zeta}_j + (1-\lambda')/2}{\tilde{\rho}_{\mathrm{L}} + \lambda' \check{\zeta}_j - (1-\lambda')/2} \right)^{1-\delta_{\sigma_j \tau_{\mathrm{L}}}}$$
$$\times \prod_{j \in I} \left(\frac{-\tilde{\rho}_{\mathrm{R}} - \lambda' \check{\zeta}_j + (1-\lambda')/2}{-\tilde{\rho}_{\mathrm{R}} - \lambda' \check{\zeta}_j - (1-\lambda')/2} \right)^{1-\delta_{\sigma_j \tau_{\mathrm{R}}}}$$
$$\times \prod_{\substack{j<k \\ j \in J, k \in I}} \left(\frac{1}{2\lambda'} + \frac{\check{\zeta}_j - \check{\zeta}_k}{2} \right)^{1-\delta_{\sigma_j\sigma_k}} \left(\frac{\check{\zeta}_j - \check{\zeta}_k}{2} \right)^{-\delta_{\sigma_j\sigma_k}}$$
$$\times \prod_{\substack{j<k \\ j \in I, k \in J}} \left(\frac{-1}{2\lambda'} + \frac{\check{\zeta}_j - \check{\zeta}_k}{2} \right)^{1-\delta_{\sigma_j\sigma_k}} \left(\frac{\check{\zeta}_j - \check{\zeta}_k}{2} \right)^{-\delta_{\sigma_j\sigma_k}}. \tag{93}$$

4. Thermodynamic Limit

The thermodynamic limit of the two parts, 2 quasi-particles and λ' quasi-holes state and 1 quasi-particles state, must be taken separately. To take the thermodynamic limit of two quasi-particle part, the thermodynamic limit of M cannot be taken straightforwardly, for the reason that the first several highest orders from M_ν are

cancelled through the sum over μ. The leading terms from M are of the order $N^{-(\lambda+1)}$. The proof is shown below.

The columns with $\zeta'_j - \mu'_j = 0$ ($j \in [1, \lambda']$) are labelled by I, and the other columns are labelled by J. New variables $e_j = 1 (j \in J), -1 (j \in I)$ are introduced. With these variables, M_ν is rewritten as

$$M = \prod_{\substack{j<k \\ j,k \in P_\nu \mathrm{or} Q_\nu}} \left(\frac{\check{\zeta}_j - \check{\zeta}_k}{2} \right)^{-1} \sum_{\substack{\mu \in \Lambda_N \\ \mathrm{s.t.} \nu/\mu:\mathrm{h.s.} \\ n_\mu = \lambda, \lambda+1,\ \mu_1 \leq \lambda'}} \prod_{j:\mathrm{odd}}^{\lambda'} e_j \prod_{\substack{j<k \\ j,k \in P_\mu \mathrm{or} Q_\mu}} \left(\frac{\check{\zeta}_j - \check{\zeta}_k}{2} \right)$$

$$\times \prod_j \left(1 - \frac{\frac{\lambda}{\lambda'} \frac{1+e_j}{2}}{\frac{1}{2}\left(\frac{\tilde{\rho}_\mathrm{L}}{\lambda'} + \check{\zeta}_j + \frac{1}{2}\left(1 - \frac{1}{\lambda'}\right)\right)} \right)^{1-\delta_{\sigma_j \tau_\mathrm{L}}}$$

$$\left(1 - \frac{\frac{\lambda}{\lambda'} \frac{1-e_j}{2}}{\frac{1}{2}\left(-\frac{\tilde{\rho}_\mathrm{R}}{\lambda'} - \check{\zeta}_j - \frac{1}{2}\left(1 - \frac{1}{\lambda'}\right)\right)} \right)^{1-\delta_{\sigma_j \tau_\mathrm{R}}}$$

$$\times \prod_{j<k} \left(1 + \frac{\frac{1}{2\lambda'} \frac{e_j - e_k}{2}}{\frac{\check{\zeta}_j - \check{\zeta}_k}{2}} \right)^{1-\delta_{\sigma_j \sigma_k}} \tag{94}$$

Since e_j always appears together with N^{-1} as e_j/N, expansion with respect to N^{-1} is equivalent to that with e_j. Except for the first row in the right-hand side of (94), which is independent of μ, M has the following form:

$$\sum_{\substack{\mu \in \Lambda_N \\ \mathrm{s.t.} \nu/\mu:\mathrm{h.s.} \\ n_\mu = \lambda, \lambda+1,\ \mu_1 \leq \lambda'}} \prod_{j<k} (u_k - u_j)^{\delta_{\sigma^\mu_j \sigma^\mu_k}} \prod_{j:\mathrm{odd}} e_j \cdot (1 + A_1 e_1 + \cdots). \tag{95}$$

Here σ^μ_j is defined as 1/2 (-1/2) when $\mu'_j - j$ is odd (even). Then the problem reduces to finding the set S satisfying

$$\sum_\mu \prod_{j<k} (u_k - u_j)^{\delta_{\sigma^\mu_j \sigma^\mu_k}} \prod_{j \in S} e_j \neq 0. \tag{96}$$

These sets of S are given by

$$\{ S \subset [1, \lambda'] \,|\, \sharp\{S \cap \mathrm{odd}\} = \sharp\{S \cap \mathrm{even}\} \}. \tag{97}$$

This relation is proved in Appendix B. For the term proportional to $\prod_{j \in Q} e_j$ in the expansion $(1 + A_1 e_1 + \cdots)$, Q is related to S as

$$\prod_{j:\mathrm{odd}} e_j \prod_{j \in Q} e_j = \prod_{j \in S} e_j, \tag{98}$$

and Q is equivalent to the set Q_S in (B.12). Thus the set of Q that contribute to M in the highest order is given by

$$\{ Q \subset [1, \lambda'] \,|\, \sharp\{Q\} = \lambda + 1 \}. \tag{99}$$

For the Q satisfying (99), the coefficients of $\prod_{j\in Q} e_j$ in $(1 + A_1 e_1 + \cdots)$ is given by

$$\left(\frac{1}{2\lambda'}\right)^{\lambda+1} \tilde{W}^{-1} \prod_{j\in Q} \frac{\partial}{\partial \nu'_j} \tilde{W} \tag{100}$$

with

$$\begin{aligned}\tilde{W} &= \prod_{j<k} \left(\frac{\check{\zeta}_j - \check{\zeta}_k}{2}\right)^{1-\delta_{\sigma_j\sigma_k}} \\ &\times \prod_{j=1}^{\lambda'} \left(\frac{1}{2}\left(\frac{\tilde{\rho}_{\mathrm{L}}}{\lambda'} + \check{\zeta}_j + \frac{1}{2}\left(1 - \frac{1}{\lambda'}\right)\right)\right)^{-2\lambda(1-\delta_{\sigma_j\tau_{\mathrm{L}}})} \\ &\times \left(-\frac{1}{2}\left(\frac{\tilde{\rho}_{\mathrm{R}}}{\lambda'} + \check{\zeta}_j + \frac{1}{2}\left(1 - \frac{1}{\lambda'}\right)\right)\right)^{-2\lambda(1-\delta_{\sigma_j\tau_{\mathrm{R}}})} .\end{aligned} \tag{101}$$

The energy and momentum for ν in the thermodynamic limit is given by

$$\begin{aligned}\omega_\nu &= E_{N+1}(\kappa) - E_N(\mathrm{g}) \\ &= \lambda' \left(\frac{\pi d}{2}\right)^2 \left[\lambda'(w_1^2 + w_2^2) - \sum_{j=1}^{\lambda'} u_j^2\right], \\ P_\nu &= -\frac{\pi d}{2}\left[\lambda'(w_1 + w_2) + \sum_{j=1}^{\lambda'} u_j\right].\end{aligned}$$

To take the thermodynamic limit, the sum over μ is replaced by integrals about momenta of quasi-holes and quasi-particles and the sums over spins of them.

$$\begin{aligned}\frac{1}{L(N+1)} \sum_{\nu\in\mathcal{L}_N} \rightarrow\ & \lambda'^2 \left(\frac{N}{4}\right)^{2\lambda+3} \sum_{\tau_{\mathrm{L}},\tau_{\mathrm{R}}} \int_1^\infty \mathrm{d}w_1 \int_{-\infty}^{-1} \mathrm{d}w_2 \\ & \times \prod_{j=1}^{\lambda'} \left[\sum_{\sigma_j} \int_{u_{j+1}}^{u_{j-1}} \mathrm{d}u_j\right] \delta_{\sum \sigma_j + \tau_{\mathrm{L}} + \tau_{\mathrm{R}}, -1/2}.\end{aligned} \tag{102}$$

Particle propagator in the thermodynamic limit is written as the sum of two parts, one has two quasi-particles and λ' quasi-holes moving each area in the dispersion, the other has one quasi-particle moving left-side of left Fermi point or right-side of right Fermi point with λ' quasi-holes residing at same Fermi point,

$$G^+(x,t) = G^{+0}(x,t) + G^{+1}(x,t). \tag{103}$$

For one quasi-particle excitation state, $\nu'_j - \mu'_j$ can take only 0 for right-moving quasi-particle, and 1 for left-moving one, and thus one quasi-particle states must be separated from two quasi-particles states. Each term for the above two situations is given as follows:

$$\begin{aligned}G^{+0}(x,t) = C^0 \int_1^\infty \mathrm{d}w \left(\frac{w-1}{w+1}\right)^\lambda \exp\left[-\mathrm{i}\left(\frac{\pi d}{2}\right)^2 \lambda'^2 w^2 t\right] \\ \times \cos\left[\frac{\pi d}{2}\lambda' w x\right]\end{aligned}$$

$$G^{+1}(x,t) = C^1 \sum_{\tau_{\rm L},\tau_{\rm R}} \int_1^\infty \mathrm{d}w_1 \int_{-\infty}^{-1} \mathrm{d}w_2 \prod_{j=1}^{\lambda'} \left[\sum_{\sigma_j} \int_{u_{j+1}}^{u_{j-1}} \mathrm{d}u_j \right] \times \delta_{\sum \sigma_j + \tau_{\rm L} + \tau_{\rm R}, -1/2} F(u,w)\, e^{-\mathrm{i}\omega_\nu t + \mathrm{i} P_\nu x} \tag{104}$$

$$C^0 = \frac{\lambda' d}{2}$$

$$C^1 = \frac{2^{2(\lambda+2)} d}{\lambda'^\lambda} \frac{1}{\Gamma(\lambda+1)} \prod_{j=1}^{\lambda'} \frac{\Gamma\left((\lambda+1)/\lambda'\right)}{\Gamma\left(j/\lambda'\right)^2} \tag{105}$$

$$F(u,w) = \frac{(w_1^2-1)^\lambda (w_2^2-1)^\lambda}{(w_1-w_2)^{2\lambda}} \prod_{j=1}^{\lambda'} (1-u_j^2)^{-\lambda/\lambda'} \prod_{j<k} (u_k-u_j)^{-2\lambda/\lambda'} \times \left(\sum_{\substack{Q\in[1,\lambda'] \\ \text{s.t.}\, |Q|=\lambda+1}} (-1)^{|(\bar{Q}\cap Q_\nu \cap \mathrm{odd}) \cup (Q \cap \bar{Q}_\nu \cap \mathrm{odd})|} \prod_{\substack{j<k \\ j,k\in Q(\bar{Q})}} (u_k-u_j) \cdot W_\sigma^{-1} \prod_{j\in Q} \frac{\partial}{\partial u_j} W_\sigma \right)^2 \tag{106}$$

$$W_\sigma = \prod_{j<k} (u_k-u_j)^{1-\delta_{\sigma_j\sigma_k}} \prod_{j=1}^{\lambda'} (w_1-u_j)^{-2\lambda(1-\delta_{\sigma_j\tau_{\rm L}})} (u_j-w_2)^{-2\lambda(1-\delta_{\sigma_j\tau_{\rm R}})}. \tag{107}$$

5. Spectral Function

Fourier transformation of single particle Green's function gives a spectral function of spin 1/2 Calogero-Sutherland model. The knowledge about the analytical representation of of the thermodynamic limit of particle propagator together with hole propagator lead to a spectral function in whole range of an energy-momentum space. The spectral function in lower half plane of an energy momentum plane is deduced from hole propagator with reversed position and time

$$A^-(p,\epsilon) = \frac{1}{2\pi} \int_{-\infty}^\infty \mathrm{d}x \int_{-\infty}^\infty \mathrm{d}t\, e^{\mathrm{i}(\epsilon-\mu)t - \mathrm{i}Px} \frac{\langle \mathrm{g}, N | \psi_\downarrow^\dagger(0,0) \psi_\downarrow(x,t) | \mathrm{g}, N \rangle}{\langle \mathrm{g}, N | \mathrm{g}, N \rangle}, \tag{108}$$

and that in upper half plane is from particle propagator

$$A^+(p,\epsilon) = \frac{1}{2\pi} \int_{-\infty}^\infty \mathrm{d}x \int_{-\infty}^\infty \mathrm{d}t\, e^{\mathrm{i}(\epsilon-\mu)t - \mathrm{i}Px} G^+(x,t). \tag{109}$$

There are four power-law divergent lines about the intensity of the spectral function inside the support on the upper-half plane. From the elementary excitation point of view, each line can be interpreted as the excitation of single quasi-particle out of the Fermi surface with the other elementary excitations fixed at the two Fermi points. Besides the internal lines, the boundaries of the support of the upper-half plane have the delta function shaped divergent raised from G^{0L} and G^{0R}. 0L and 0R states are distinguished from the internal divergent states regarding the distribution of quasi-holes at Fermi points.

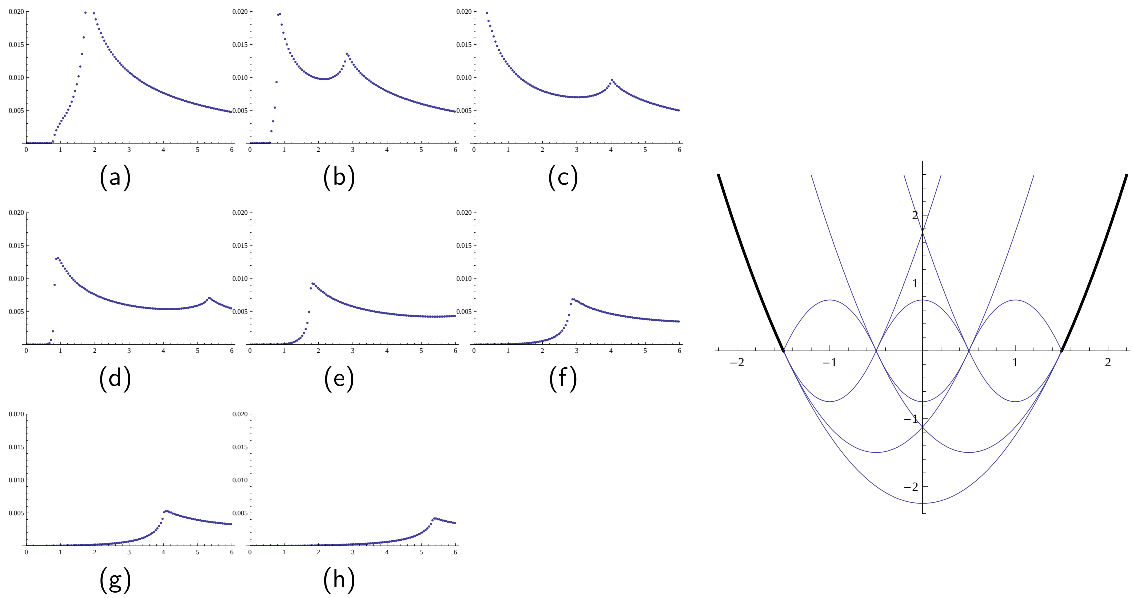

Figure 1. Spectral function of spin 1/2 Calogero-Sutherland model for $\lambda = 1$. The upper-half of an energy-momentum plane corresponds to the spectral function of the particle propagator, and the excitation of 3 quasi-holes and 2 quasi-particles exhaust the whole range of the support in the upper-half plane.

All the divergent lines in the upper-half plane are continuously connected to those in the lower-half lines, for the two of eight single quasi-particle excitation states lines in the upper-half plane with no corresponding lines in the lower-half plane have no singularities about the intensity. The support at low energy has the characteristics of the Tomonaga-Luttinger liquid.

6. Conclusion

We calculated the particle propagator of the spin Calogero-Sutherland model for the finite-size system and the thermodynamic limit with use of the Uglov's method.

The spectral function of single-particle Green's function of spin Calogero-Sutherland model in the whole range of the energy-momentum space were obtained. Each divergent line connects continuously between upper and lower-half of the energy momentum space. The interpretation of these intensity divergent states were given with respect to the elementary excitations.

Acknowledgments

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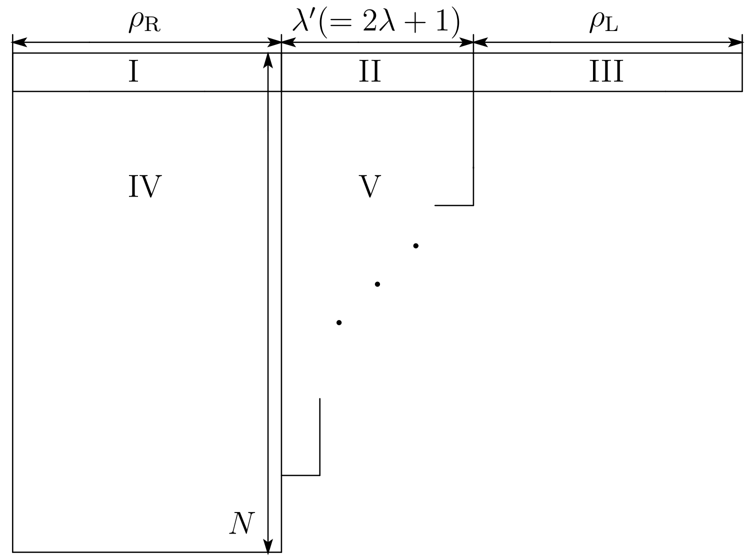

Figure A1. Decomposition of the Young diagram.

Appendix A. Details of the Finite Size Representation

In this section we derive the finite-size representation of the matrix element. The terms peculiar to the particle propagator are $\psi^{(\lambda')}_{\nu\mu}$ and the norm of ν in (36). In calculating the norm of ν, it is sufficient to obtain the difference from $\tilde{\nu}$, the quasi-hole part of ν. To calculate them systematically, the Young diagram is decomposed into five parts as shown in Fig.A1

Appendix A.1. the norm of ν

For the later convenience, we set

$$\tilde{\nu}=(\lambda',\nu_2,\cdots,\nu_N,0)=(\lambda',\zeta_1,\cdots,\zeta_{N-1},0)\in\Lambda_{N+1}$$

and decompose the norm $\{P^{(\lambda')}_{\nu},P^{(\lambda')}_{\nu}\}_{N+1,\lambda}$ as

$$\{P^{(\lambda')}_{\nu},P^{(\lambda')}_{\nu}\}_{N+1,\lambda}=\{P^{(\lambda')}_{\tilde{\nu}},P^{(\lambda')}_{\tilde{\nu}}\}_{N+1,\lambda}\times\mathcal{N}_{\nu\tilde{\nu}}$$

with

$$\mathcal{N}_{\nu\tilde{\nu}}=\frac{\{P^{(\lambda')}_{\nu},P^{(\lambda')}_{\nu}\}_{N+1,\lambda}}{\{P^{(\lambda')}_{\tilde{\nu}},P^{(\lambda')}_{\tilde{\nu}}\}_{N+1,\lambda}}$$

First we consider $\{P^{(\lambda')}_{\tilde{\nu}},P^{(\lambda')}_{\tilde{\nu}}\}_{N+1,\lambda}$, which is given by

$$\frac{c^{(\lambda',2)}_{N+1}Y'_{\tilde{\nu}}(1)Z_{\tilde{\nu}}(1/(2\lambda'))}{Y'_{\tilde{\nu}}((\lambda'+1)/(2\lambda'))Z_{\tilde{\nu}}(1/2)}. \tag{A.1}$$

$Z_{\tilde{\nu}}(r)$ is decoupled into the contribution from the first row in $D(\tilde{\nu})$ and other rows. In the first row, $a(s)=\lambda'-j$ and $l(s)=\zeta'_j$ and hence $s\in H_2(\tilde{\nu})$ in the first row is given

by $s=(1,j)$ with $\sigma_j=-1/2$. We thus obtain

$$\prod_{s=(1,j)\in H_2(\tilde{\nu})}\left(\frac{a(s)}{2\lambda'}+\frac{l(s)}{2}+r\right)=\prod_{j=1}^{\lambda'}\left(\frac{\zeta'_j+1}{2}-\frac{j}{2\lambda'}+r\right)^{\delta(\sigma_j,\downarrow)}$$
$$=\prod_{j=1}^{\lambda'}\left(\frac{\check{\zeta}_j-\check{\zeta}_{\lambda'+1}-1/\lambda'}{2}+r\right)^{\delta(\sigma_j,\downarrow)}. \tag{A.2}$$

The contribution from the other rows is given by $Z_\zeta(r)$ with $\zeta=(\zeta_1,\cdots,\zeta_N)\in\Lambda_N$. We obtain

$$Z_{\tilde{\nu}}(r)=Z_\zeta(r)\prod_{j=1}^{\lambda'}\left(\frac{\check{\zeta}_j-\check{\zeta}_{\lambda'+1}-1/\lambda'}{2}+r\right)^{\delta(\sigma_j,\downarrow)}. \tag{A.3}$$

The explicit expression for $Z_\zeta(r)$ is available in (96) in [25] with replacement of $\tilde{\mu'}_j$ by $\tilde{\zeta'}_j$. Consequently, we obtain (92).

$Y'_{\tilde{\nu}}(r)$, on the other hand, is decoupled into the product of the contribution from the squares $s=(\zeta'_j+1,j)$ at the bottom of each column and the remaining squares. When $s=(\zeta'_j+1,j)\in D(\tilde{\nu})\setminus C_2(\tilde{\nu})$, $\sigma_j=-1/2$ and we obtain

$$\prod_{s=(\zeta'_j+1,j)\in D(\tilde{\nu})\setminus C_2(\tilde{\nu})}\left(\frac{a'(s)}{2\lambda'}+\frac{N-1-l'(s)}{2}+r\right)$$
$$=\prod_{j=1}^{\lambda'}\left(\frac{\check{\zeta}_0-\check{\zeta}_j-1/\lambda'}{2}+r\right)^{\delta(\sigma_j,\downarrow)}. \tag{A.4}$$

The remaining contribution to $Y'_{\tilde{\nu}}(r)$ is given by $Y'_\zeta(r)$ and as a result, $Y'_{\tilde{\nu}}(r)$ is written as

$$Y'_{\tilde{\nu}}(r)=Y'_\zeta(r)\prod_{j=1}^{\lambda'}\left(\frac{\check{\zeta}_0-\check{\zeta}_j-1/\lambda'}{2}+r\right)^{\delta(\sigma_j,\downarrow)}. \tag{A.5}$$

The expression for $Y'_\zeta(r)$ has been derived in [26] as

$$Y'_\zeta(r)=\prod_{j=1}^{\lambda'}\frac{\Gamma\left[N/2+(j-1)/\lambda'+r\right]}{\Gamma\left[(\check{\zeta}_0-\check{\zeta}_j+2-1/\lambda'-\delta_{\sigma_0,\sigma_j})/2+r\right]}. \tag{A.6}$$

From (A.5) and (A.6), we obtain (91)

The norm $\{P_{\tilde{\nu}}^{(\lambda')},P_{\tilde{\nu}}^{(\lambda')}\}_{N+1,\lambda}$ is explicitly given by

$$\begin{aligned}&\{P_{\tilde{\nu}}^{(\lambda')},P_{\tilde{\nu}}^{(\lambda')}\}_{N+1,\lambda}\\&=\frac{1}{(\Gamma\left[\frac{\lambda'+1}{2\lambda'}\right])^{\lambda'}}\frac{\Gamma\left[(N+2-2/\lambda')/2+j/\lambda'\right]}{\Gamma\left[(N+1-1/\lambda')/2+j/\lambda'\right]}\\&\times\frac{\Gamma\left[(\check{\zeta}_0-\check{\zeta}_j+1+\delta_{\sigma_j,\sigma_0})/2\right]}{\Gamma\left[(\check{\zeta}_0-\check{\zeta}_j+2-1/\lambda'+\delta_{\sigma_j,\sigma_0})/2\right]}\\&\times\prod_{j<k}\frac{\Gamma\left[(\check{\zeta}_j-\check{\zeta}_k+1-\delta_{\sigma_j,\sigma_k})/2\right]\Gamma\left[(\check{\zeta}_j-\check{\zeta}_k+1+\delta_{\sigma_j,\sigma_k})/2\right]}{\Gamma\left[(\check{\zeta}_j-\check{\zeta}_k+2-1/\lambda'-\delta_{\sigma_j,\sigma_k})/2\right]\Gamma\left[(\check{\zeta}_j-\check{\zeta}_k+1/\lambda'+\delta_{\sigma_j,\sigma_k})/2\right]}.\end{aligned} \tag{A.7}$$

The expression $\mathcal{N}_{\nu,\tilde{\nu}}$ is written as

$$\mathcal{N}_{\nu\tilde{\nu}} = \frac{\{P^{(\lambda')}_{\nu_+}, P^{(\lambda')}_{\nu_+}\}_{N+1,\lambda}}{\{P^{(\lambda')}_{\tilde{\nu}_+}, P^{(\lambda')}_{\tilde{\nu}_+}\}_{N+1,\lambda}}$$

with

$$\begin{aligned} \nu_+ &= (\nu_1 + \rho_{\mathrm{R}}, \cdots, \nu_{N+1} + \rho_{\mathrm{R}}) \\ &= (\lambda' + \rho_{\mathrm{L}} + \rho_{\mathrm{R}}, \zeta_1 + \rho_{\mathrm{R}}, \cdots, \zeta_{N-1} + \rho_{\mathrm{R}}, 0) \in \Lambda_{N+1} \end{aligned} \tag{A.8}$$

and

$$\begin{aligned} \tilde{\nu}_+ &= (\tilde{\nu}_1 + \rho_{\mathrm{R}}, \cdots, \tilde{\nu}_{N+1} + \rho_{\mathrm{R}}) \\ &= (\lambda' + \rho_{\mathrm{R}}, \zeta_1 + \rho_{\mathrm{R}}, \cdots, \zeta_{N-1} + \rho_{\mathrm{R}}, \rho_{\mathrm{R}}) \in \Lambda_{N+1}. \end{aligned} \tag{A.9}$$

Let $\mathcal{N}_{\nu,\tilde{\nu}}$ be decoupled into the following five parts:

(I) the contribution $\mathcal{N}^{(\mathrm{I})}$ from $s = (1, j)$ with $j \in [1, \rho_{\mathrm{R}}]$.

(II) the contribution $\mathcal{N}^{(\mathrm{II})}$ from $s = (1, j)$ with $j \in [\rho_{\mathrm{R}} + 1, \rho_{\mathrm{R}} + \lambda']$.

(III) the contribution $\mathcal{N}^{(\mathrm{III})}$ from $s = (1, j)$ with $j \in [\lambda' + \rho_{\mathrm{R}} + 1, \lambda' + \rho_{\mathrm{L}} + \rho_{\mathrm{R}}]$.

(IV) the contribution $\mathcal{N}^{(\mathrm{IV})}$ from $s = (i, j)$ with $i \in [2, N + 1]$ and $j \in [1, \rho_{\mathrm{R}}]$.

(V) the contribution $\mathcal{N}^{(\mathrm{V})}$ from $s = (i, j)$ with $i \in [2, \zeta'_1 + 1]$ and $j \in [\rho_{\mathrm{R}} + 1, \rho_{\mathrm{R}} + \lambda']$.

See also Fig. A1.

The contribution $\mathcal{N}^{(\mathrm{I})}$ comes from the $Z_\nu(1/(2\lambda'))/Z_\nu(1/2)$ divided by $Z_{\tilde{\nu}}(1/(2\lambda'))/Z_{\tilde{\nu}}(1/2)$ and is explicitly expressed as

$$\begin{aligned} \mathcal{N}^{(\mathrm{I})} = &\frac{\Gamma\left(\lambda'(N+2)/2\right)}{\Gamma\left[(1 + \lambda'(N+1))/2\right]} \frac{\Gamma\left((\rho_{\mathrm{L}} + \lambda' N + 1 + \delta_{\tau_1,\uparrow})/2\right)}{\Gamma\left((\rho_{\mathrm{L}} + \lambda'(N+1) + \delta_{\tau_1,\uparrow})/2\right)} \\ &\frac{\Gamma\left((\rho_{\mathrm{R}} + 1 + \lambda'(N+1) + \delta_{\tau_2,\uparrow})/2\right)}{\Gamma\left((\rho_{\mathrm{R}} + \lambda'(N+2) + \delta_{\tau_2,\uparrow})/2\right)} \frac{\Gamma\left((\rho_{\mathrm{L}} + \rho_{\mathrm{R}} + \lambda'(N+1) + 1 - \delta_{\tau_1,\tau_2})/2\right)}{\Gamma\left((\rho_{\mathrm{L}} + \rho_{\mathrm{R}} + \lambda' N + 2 - \delta_{\tau_1,\tau_2})/2\right)} \end{aligned} \tag{A.10}$$

Similarly, $\mathcal{N}^{(\mathrm{II})}$ and $\mathcal{N}^{(\mathrm{III})}$ are expressed as

$$\mathcal{N}^{(\mathrm{II})} = \prod_j \left(\frac{-j + 1 + \lambda'\nu'_j}{-j + \lambda'(\nu'_j + 1)}\right)^{1-\delta_{\sigma_j\uparrow}} \left(\frac{\rho_{\mathrm{L}} - j + \lambda'(\nu'_j + 1)}{\rho_{\mathrm{L}} + 1 - j + \lambda'\nu'_j}\right)^{1-\delta_{\sigma_j\tau_{\mathrm{L}}}} \tag{A.11}$$

and

$$\begin{aligned} \mathcal{N}^{(\mathrm{III})} = &\frac{\Gamma\left((\rho_{\mathrm{L}} + \lambda' + \delta_{\tau_1,\uparrow})/2)\right)}{\Gamma\left((1 + \lambda')/2\right)\Gamma\left((\rho_{\mathrm{L}} + 1 + \delta_{\tau_1,\uparrow})/2)\right)} \\ &\frac{\Gamma\left((\rho_{\mathrm{R}} + \lambda'(N+2) + \delta_{\tau_2,\uparrow})/2\right)}{\Gamma\left((\rho_{\mathrm{R}} + \lambda'(N+1) + 1 + \delta_{\tau_2,\uparrow})/2\right)} \frac{\Gamma\left((\rho_{\mathrm{L}} + \rho_{\mathrm{R}} + \lambda'(N+1) + 1 + \delta_{\tau_1,\tau_2})/2\right)}{\Gamma\left((\rho_{\mathrm{L}} + \rho_{\mathrm{R}} + \lambda'(N+2) + \delta_{\tau_1,\tau_2})/2\right)} \end{aligned} \tag{A.12}$$

The contribution to $\mathcal{N}^{(\mathrm{IV})}$ consists of $Z_\nu(1/(2\lambda'))/Z_\nu(1/2)$ divided by $Z_{\tilde{\nu}}(1/(2\lambda'))/Z_{\tilde{\nu}}(1/2)$ from ith row with $i \in [2, N]$ and $Y'_{\tilde{\nu}}(1/2 + 1/(2\lambda'))/Y'_{\tilde{\nu}}(1)$;

$$\mathcal{N}^{(\mathrm{IV})} = \prod_j \left(\frac{j + \lambda'(N - \nu'_j)}{j - 1 + \lambda'(N - \nu'_j + 1)}\right)^{\delta_{\sigma_j\uparrow}} \left(\frac{\rho_{\mathrm{R}} + j - 1 + \lambda'(N - \nu'_j + 1)}{\rho_{\mathrm{R}} + j + \lambda'(N - \nu'_j)}\right)^{\delta_{\sigma_j,\tau_2}}$$

$$\times \frac{\Gamma\left((\rho_{\mathrm{R}}+\lambda' N+1+\delta_{\tau_2,\downarrow})\right)/2)}{\Gamma\left((\rho_{\mathrm{R}}+\lambda'(N+1)+\delta_{\tau_2,\downarrow})\right)/2)} \frac{\Gamma\left((1+\lambda'(N+1))/2\right)}{\Gamma\left((2+\lambda' N)/2\right)}$$
$$\times \frac{\Gamma\left((\rho_{\mathrm{R}}+\lambda'+\delta_{\tau_2,\downarrow})/2\right)}{\Gamma\left((1+\lambda')/2\right)\Gamma\left((\rho_{\mathrm{R}}+1+\delta_{\tau_2,\downarrow})/2\right)}. \tag{A.13}$$

We easily see that $\mathcal{N}_V = 1$ because the contribution from $D(\nu_+)$ exactly cancels with that from $D(\tilde{\nu}_+)$.

Appendix A.2. $\psi^{(\lambda')}_{\nu,\mu}$

When $\rho_{\mathrm{R}} > 0$, the columns satisfying $k \in C_{\nu/\mu}$ are equivalent to $j \in J$ for $j \in [1, \lambda']$, and $j \in [-\rho_{\mathrm{R}}+1, 0]$.

When $\psi^{(\lambda')}_{\nu,\mu}$ is decoupled into four parts

$$\psi^{(\lambda')}_{\nu,\mu} = V^{(\mathrm{I})}_{\nu,\mu} V^{(\mathrm{II})}_{\nu,\mu} V^{(\mathrm{IV})}_{\nu,\mu} V^{(\mathrm{V})}_{\nu,\mu},$$

each part is given, respectively, by

$$V^{(\mathrm{I})}_{\nu,\mu} = \frac{\Gamma\left(1+\lambda' N/2\right)\Gamma\left[(\tilde{\rho}_{\mathrm{L},0}-\tilde{\rho}_{\mathrm{R}}+\lambda'+1-\delta_{\tau_{\mathrm{R}},\uparrow})/2\right]}{\Gamma\left((1+\lambda'(N+1))/2\right)\Gamma\left[(\tilde{\rho}_{\mathrm{L},0}-\tilde{\rho}_{\mathrm{R}}+2-\delta_{\tau_{\mathrm{R}},\uparrow})/2\right]}$$
$$\times \frac{\Gamma\left[(\tilde{\rho}_{\mathrm{L}}-\tilde{\rho}_{\mathrm{R},0}+\lambda'+\delta_{\tau_{\mathrm{L}},\uparrow})/2\right]\Gamma\left[(\tilde{\rho}_{\mathrm{L}}-\tilde{\rho}_{\mathrm{R}}+2-\lambda'-\delta_{\tau_1,\tau_2})/2\right]}{\Gamma\left[\tilde{\rho}_{\mathrm{L}}-\tilde{\rho}_{\mathrm{R},0}+1+\delta_{\tau_{\mathrm{L}},\uparrow})/2\right]\Gamma\left[(\tilde{\rho}_{\mathrm{L}}-\tilde{\rho}_{\mathrm{R}}+1-\delta_{\tau_1,\tau_2})/2\right]} \tag{A.14}$$

$$V^{(\mathrm{II})}_{\nu,\mu} = \prod_{j\in J} \left(\frac{\check{\zeta}_j-\check{\zeta}_{\lambda'+1}+1-1/\lambda'}{\check{\zeta}_j-\check{\zeta}_{\lambda'+1}}\right)^{1-\delta_{\sigma_j\uparrow}} \left(\frac{\tilde{\rho}_{\mathrm{L}}+\lambda'\check{\zeta}_j+(1-\lambda')/2}{\tilde{\rho}_{\mathrm{L}}+\lambda'\check{\zeta}_j-(1-\lambda')/2}\right)^{1-\delta_{\sigma_j\tau_{\mathrm{L}}}} \tag{A.15}$$

$$V^{(\mathrm{IV})}_{\nu,\mu} = \prod_{j\in I} \frac{\Gamma\left[(\lambda'(\check{\zeta}_0-\check{\zeta}_j)+1-\delta_{\sigma_j\uparrow})/2\right]\Gamma\left[(\lambda'(\check{\zeta}_0-\check{\zeta}_j+1)+\delta_{\sigma_j\uparrow})/2\right]}{\Gamma\left[(\lambda'(\check{\zeta}_0-\check{\zeta}_j)+1+\delta_{\sigma_j\uparrow})/2\right]\Gamma\left[(\lambda'(\check{\zeta}_0-\check{\zeta}_j+1)-\delta_{\sigma_j\uparrow})/2\right]}$$
$$\times \frac{\Gamma\left[(-\tilde{\rho}_{\mathrm{R}}-\lambda'\check{\zeta}_j+(\lambda'-1)/2+\delta_{\sigma_j,\tau_2})/2\right]}{\Gamma\left[(-\tilde{\rho}_{\mathrm{R}}-\lambda'\check{\zeta}_j+(\lambda'+3)/2-\delta_{\sigma_j,\tau_2})/2\right]}$$
$$\times \frac{\Gamma[(-\tilde{\rho}_{\mathrm{R}}-\lambda'\check{\zeta}_j+(5-\lambda')/2-\delta_{\sigma_j,\tau_2})/2]}{\Gamma\Gamma[(-\tilde{\rho}_{\mathrm{R}}-\lambda'\check{\zeta}_j+(1-\lambda')/2+\delta_{\sigma_j,\tau_2})/2]} \tag{A.16}$$

$$V^{(\mathrm{V})}_{\nu,\mu} = \prod_{\substack{j<k \\ \text{s.t.}\, j\in J, k\in I}} \left(\frac{\check{\zeta}_j-\check{\zeta}_k+1-1/\lambda'}{\check{\zeta}_j-\check{\zeta}_k}\right)^{\delta_{\sigma_j\sigma_k}} \left(\frac{\check{\zeta}_j-\check{\zeta}_k+1/\lambda'}{\check{\zeta}_j-\check{\zeta}_k+1}\right)^{1-\delta_{\sigma_j\sigma_k}} \tag{A.17}$$

Appendix B. Proof of (97)

Here we prove that the subset S of $[1, \lambda']$ satisfying

$$\{S \subset [1,\lambda'] \,|\, \sharp\{S\cap \mathrm{odd}\} = \sharp\{S \cap \mathrm{even}\}\} \tag{B.1}$$

gives nonzero value in

$$\sum_{\substack{\mu\in\Lambda_N\\ \text{s.t.}\nu/\mu:\text{h.s.}\\ n_\mu=\lambda,\lambda+1,\ \mu_1\le\lambda'}} \prod_{j<k}(\check{\zeta}_j-\check{\zeta}_k)^{\delta_{\sigma^\mu_j\sigma^\mu_k}}\prod_{j\in S}\alpha_j. \tag{B.2}$$

For the case that S has odd elements, any partition with an $I=I_0$ has a counterpart partition which has $I=\bar{I}_0$. Since $\prod_{j\in S}\alpha_j$ with I_0 and that with $\bar{I}_0$ has an opposite sign, and both has the same term about $\prod_{j<k}(\check{\zeta}_j-\check{\zeta}_k)^{\delta_{\sigma^\mu_j\sigma^\mu_k}}$, contribution from those two partition are cancelled. Thus, in the following, the case that S has even elements are considered. In this case, since the sum about $n_\mu=\lambda$ and $n_\mu=\lambda+1$ gives the same contribution, we only consider the sum about $n_\mu=\lambda+1$ and finally twice the result is equal to (B.2).

Expanding the product $\prod_{j<k,j,k\in A}(u_j-u_k)$ for a specific set A, there appear $|A|!$ terms in which each term is the product of u_i with no same index as

$$\prod_{\substack{j<k\\ j,k\in A}}(u_j-u_k)=\sum_{\sigma\in S_{|A|}}(-1)^\sigma\prod_i u^{i-1}_{\sigma(i)}, \tag{B.3}$$

where $(-1)^\sigma$ is the sign of a permutation σ with $(-1)^\sigma=1$ for $\sigma(i)=|A|+1-i$.

We introduce the set Q, whose number of the element of is $\lambda+1$. The difference of the sign of the term $u^0_{a_0}u^1_{a_1}\cdots u^{\lambda-1}_{a_{\lambda-1}}u^0_{b_0}\cdots u^{\lambda-1}_{b_{\lambda-1}}u^\lambda_{b_\lambda}$ between one from

$$\prod_{\substack{j<k\\ j,k\in Q\text{or}\bar{Q}}}(u_j-u_k)\prod_{j\in S}\alpha_j. \tag{B.4}$$

with $Q=\{b_0,b_1,\cdots,b_\lambda\}$ and with $Q'=\{b_0,\cdots,b_{i-1},a_i,b_{i+1},\cdots,b_\lambda\}$ is given by the product of $(-1)^{\sharp\{a_i,b_i|a_i,b_i\in S\}}$ (permuting a_i and b_i results in changing a_i or $b_i\in I$ with a_i or $b_i\in J$), $(-1)^{(\sigma_Q-\sigma_{Q'})+(\sigma_{\bar{Q}}-\sigma_{\bar{Q}'})}$, where $\bar{A}$ means the complementary set of A. To be explicitly describe the each case, one can find that

$$a_i,b_i\notin S\ \Rightarrow\ (-1)^{a_i-b_i-1} \tag{B.5}$$

$$a_i\in S,b_i\notin S \text{ or } a_i\notin S,b_i\in S\ \Rightarrow\ (-1)^{a_i-b_i} \tag{B.6}$$

$$a_i,b_i\in S\ \Rightarrow\ (-1)^{a_i-b_i-1}. \tag{B.7}$$

Thus the condition that $u^0_{a_0}u^1_{a_1}\cdots u^{\lambda-1}_{a_{\lambda-1}}u^0_{b_0}\cdots u^{\lambda-1}_{b_{\lambda-1}}u^\lambda_{b_\lambda}$ appears after the sum about P is given by

$$\begin{aligned} &a_i,b_i\notin S\ \Rightarrow\ a_i-b_i:\text{odd}\\ &a_i\in S,b_i\notin S \text{ or } a_i\notin S,b_i\in S\ \Rightarrow\ a_i-b_i:\text{even}\\ &a_i,b_i\in S\ \Rightarrow\ a_i-b_i:\text{odd}\end{aligned} \tag{B.8}$$

Such terms are multiplied by $2^{\lambda+1}$ after the sum over μ, 2^λ comes from the sum over μ with $n_\mu=\lambda+1$ and the same amount from $n_\mu=\lambda$.

For $|S|=0$, all the terms in (B.4) with $Q_\emptyset = 1,3,\cdots,\lambda'$ satisfy above conditions, and we obtain

$$\sum_{\substack{\mu\in\Lambda_N \\ \text{s.t.}\nu/\mu:\text{h.s.} \\ n_\mu=\lambda,\lambda+1,\ \mu_1\le\lambda'}} \prod_{j<k} (\check{\zeta}_j-\check{\zeta}_k)^{\delta_{\sigma_j^\mu\sigma_k^\mu}} = 2^{\lambda+1} \prod_{\substack{j<k \\ j,k\in Q_\emptyset \text{or} \bar{Q}_\emptyset}} (\check{\zeta}_j-\check{\zeta}_k). \tag{B.9}$$

For $|S|=2$ and $S=l,m$, if $(l,m)=$ (even,even) or (odd,odd), there are no set satisfying (B.8). When $(l,m)=$ (even,odd), we denote the set made by exchanging l and m in $Q_\emptyset$ by Q_S. Then all the terms in (B.4) with Q_S satisfy above conditions, and

$$\sum_{\substack{\mu\in\Lambda_N \\ \text{s.t.}\nu/\mu:\text{h.s.} \\ n_\mu=\lambda,\lambda+1,\ \mu_1\le\lambda'}} \prod_{j<k} (\check{\zeta}_j-\check{\zeta}_k)^{\delta_{\sigma_j^\mu\sigma_k^\mu}} = 2^{\lambda+1}(-1)^{I_S\wedge S} \prod_{\substack{j<k \\ j,k\in Q_S \text{or} \bar{Q}_S}} (\check{\zeta}_j-\check{\zeta}_k), \tag{B.10}$$

where I_S is I for Q_S

$$I_S = \{j\in[1,\lambda']|\ j\in(Q_\nu\cap\bar{Q}_S)\cup(\bar{Q}_\nu\cap Q_S)\} \tag{B.11}$$

Here we denote Q_ν as the set of quasi-holes with spin $+1/2$. For $|S|=2n(n\in[1,\lambda])$, (B.10) can be applied by taking

$$Q_S = (S\cap\text{even})\cup(\bar{S}\cap\text{odd}) \tag{B.12}$$

for the set S satisfying $\sharp(S\cap\text{even})=\sharp(S\cap\text{odd})$. To describe sign $(-1)^{I_S\wedge S}$ by Q_ν and Q_S, S is related to Q_S as $S=(\bar{Q}_S\cap\text{odd})\cup(Q_S\cap\text{even})$, thus

$$(-1)^{I_S\cap S} = (-1)^{(\bar{Q}_S\cap Q_\nu\cap\text{odd})\cup(Q_S\cap\bar{Q}_\nu\cap\text{odd})}. \tag{B.13}$$

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